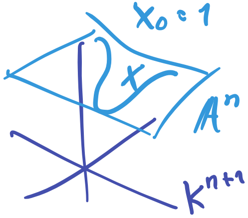
# Proj closure

**Thm** $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ aav

$I = \mathbb{I}_a(X)$

$\Rightarrow \overline{X} = Z_p(I_h) \subseteq \mathbb{P}^n$



Pf. $\boxed{\subseteq}$ you

$\boxed{\supseteq}$ Say $G \in \mathbb{I}_p(\overline{X})$

$G \in k[x_0,\dots,x_n]$ homog.

$\Rightarrow G = 0$ on $(\overline{X} \cap U_0) = X$

$\nwarrow x_0 \neq 0.$

$\Rightarrow g = G|_{x_0=1}$ is $0$ on $X$

$g \in k[x_1,\dots,x_n]$

$\Rightarrow g \in \mathbb{I}_a(X) = I$

$\Rightarrow g_h \in I_h$

$\Rightarrow G = g_h x_0^t$ some $t$.

$$\begin{tikzcd}[row sep=small] G = x_0^3x_1 + x_0^2x_1x_2 + x_0^4 \\ g = x_1 + x_1x_2 + 1 \\ g_h = x_0x_1 + x_1x_2 + x_0^2 \arrow[uu, bend right, "x_0^2"'] \end{tikzcd}$$

$\Rightarrow G \in I_h$ (since $g_h \in I_h$).

Thus $\mathbb{I}_p(\overline{X}) \subseteq I_h$ since $\overline{X}$ closed.

So $Z_p(I_h) \subseteq Z_p\mathbb{I}_p(\overline{X}) = \overline{X}$ $\square$

### Example

$$X = Z(x, y - x^2) = \{0\} \leftrightarrow [\overset{z}{1} : \overset{x}{0} : \overset{y}{0}]$$

in $\mathbb{P}^2$

$$\leadsto \bar{X} = X \leftrightarrow U_z$$

$$\neq Z(x, yz - x^2) = \{[\overset{z}{1} : \overset{x}{0} : \overset{y}{0}], [0:0:1]\}$$

↖ at ∞.

**Cor.** $X = Z(f) \Rightarrow \bar{X} = Z_p(f_h)$

**Pf.** $(f) = \{fg : g \in k[x_1, \dots, x_n]\}$ $f_h$

$$\to \bar{X} \overset{\text{Thm}}{=} Z_p((fg)_h : g \in k[x_1, \dots, x_n])$$

$$= Z_p(f_h g_h : g \in k[x_1, \dots, x_n])$$

$$= Z_p(f_h) \qquad \square$$

### Cor of Proj Null

$$\left\{\begin{array}{c}\text{irred. proj vars} \\ Y \subseteq \mathbb{P}^n \\ Y \not\subseteq Z(x_0)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{irred affine vars} \\ X \subseteq \mathbb{A}^n\end{array}\right\}$$

$$\bar{X} \longleftarrow\!\shortmid \; X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$$

$$Y \longmapsto Y \cap U_0 \cong \mathbb{A}^n$$

Why you need irreducible: (Toyesh)

$$x_0 x_2 - x_1^2$$

$$x_0^2 x_2 - x_1^2 x_0$$

Pf hint: polys ↔ polys.
(homog & dehomog).

# Morphisms

Naïve defn: polyn. maps.

Example $C = Z(xz - y^2)$

$$\varphi : \mathbb{P}^1 \longrightarrow C \subseteq \mathbb{P}^2$$

$$[s:t] \longmapsto [s^2 : st : t^2]$$

- $\varphi$ is well def
- im $\varphi = C$.

(This is a Veronese map)

In $U_t$ chart, set $u = s/t$

$$u \longmapsto (u^2, u) \in U_z$$

In $U_s$: $v \longmapsto (v, v^2) \in U_x$.

These are affine morphisms.

Now for other direction...

$$\psi : C \longrightarrow \mathbb{P}^1$$

$$[x:y:z] \longmapsto \begin{cases} [x:y] & \text{on } U_x \\ [y:z] & \text{on } U_z \end{cases}$$

Defined on all of $C$: $x = z = 0 \Rightarrow y = 0$.

Well def. on $C$: $x, z \neq 0 \Rightarrow y \neq 0$ so

$$[x:y] = [yx : y^2] = [xy : xz] = [y:z]$$

On $U_x, U_z$: $\psi$ is affine morphism.

but $\psi$ is not globally polynomial.

No way to write $\psi$ as $[f_1 : f_2]$

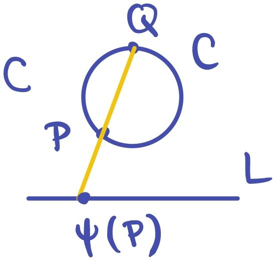
(exercise?)

# Aside: Stereographic proj.

The map $\psi$ can be defined as follows.

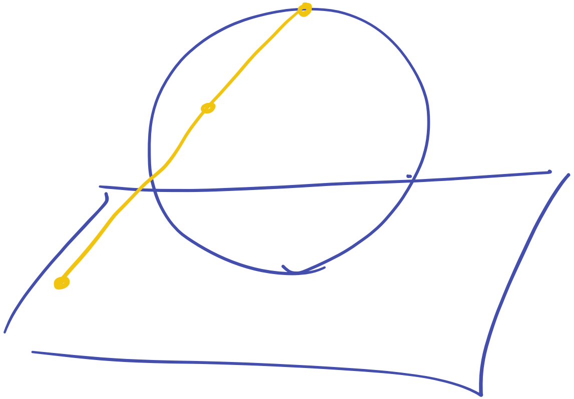
Let $Q = [\overset{x}{1} : \overset{y}{0} : \overset{z}{0}] \in C$

(pt at $\infty$)



$L = Z(x)$

line in $\mathbb{P}^2$



For $P = [a:b:c] \in C$, $P \neq Q$

The line $PQ$ is $yc = zb$

and $PQ \cap L = \psi(P) = [0:b:c]$

---

We want (need?) this to a morphism, but not a poly.