From last time

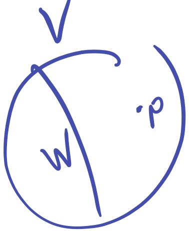
Example $C = Z(xz - y^2)$

$\varphi : \mathbb{P}^1 \to C \subseteq \mathbb{P}^2$

$[s:t] \mapsto [s^2 : st : t^2]$

$\psi : C \to \mathbb{P}^1$

$[x:y:z] \mapsto \begin{cases} [x:y] & \text{on } U_x \\ [y:z] & \text{on } U_z \end{cases}$



Today: Morphisms, birational maps

Correspondence

PAV's $\longleftrightarrow$ fg extensions of k.

Morphisms of PAVs

$V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ pav's

$f : V \to W$ is a <u>morphism</u> if

$\forall\, p \in V$ $\exists$ homog polys

$f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ s.t. for

some ~~nonempty~~ open nbd of p

$f|_U$ agrees with

$U \to \mathbb{P}^m$

$q \mapsto [f_0(q) : \cdots : f_m(q)]$

e.g. φ, ψ above.

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e.g. φ, ψ above.

Notes

1. Can also allow rat'l fns (clear denoms).
2. To have a well-def map, f_i must have same deg.
3. Also, f_i must not all vanish at a single pt.
4. Implicit: different fns agree on overlaps (since $\exists$ globally def f)

Isomorphism: if $\exists$ inverse morphism.

Examples

① φ, ψ above

$\mathbb{P}^1 \to C$ = parabola

are isomorphisms

e.g. $[s:t] \overset{\varphi}{\longmapsto} [s^2:st:t^2]$ (Assume in U_x)

$\overset{\psi|_{U_x}}{\longmapsto} [s^2:st] = [s:t]$

② Any homog rat'l fn $h: X \to k$ can be considered a morphism

$\varphi: X \to \mathbb{P}^1$. If $h = f/g$ (f, g homog same deg)

$\varphi([x_0 : \dots : x_n]) = [f(x_0,\dots,x_n) : g(x_0,\dots,x_n)]$

③ Linear change of coords on $\mathbb{P}^n$.

Later: All isoms $\mathbb{P}^n \to \mathbb{P}^n$ are of this form.

Conseq 1. H = hyperplane in $\mathbb{P}^n$

$\Longrightarrow H \cong \mathbb{P}^{n-1}$

(using: restriction of $\cong$ is $\cong$)

Conseq 2. All conics in $\mathbb{P}^2$ are $\cong$. ($k = \mathbb{C}$)

Conics $\longleftrightarrow$ symm bilin forms $\longleftrightarrow$ quad forms $\longleftrightarrow$ 3×3 matrices

$Z(x^2 + 4xy + 3y^2 + z^2)$ $\quad (x\ y\ z)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

But: All symm. ~~inv~~ $\mathbb{C}$-matrices diagable...

Coord ring of PAVs

Can define:

$$k[X] = k[x_0, \dots, x_n] / \mathbb{I}_p(X)$$

Issue #1. The elts of $k[X]$ don't give well-def fns on X.

In Fact: Every ~~poly~~ rat fn on smooth X (← defined globally) is const.

For $k = \mathbb{C}$ * this is Liouville's thm (bounded holom fns are const) plus fact that X compact.

* and X smooth.

Issue #2 Can have $X \cong Y$

$$k[X] \not\cong k[Y]$$

e.g. $X = Z_p(x) \subseteq \mathbb{P}^2 \rightsquigarrow k[X] \cong k[y,z]$ UFD

$\mathbb{P}^1 \cong X$

$Y = Z_p(x^2+y^2-z^2) \cong \mathbb{P}^1$

$\rightsquigarrow k[Y] \cong k[x,y,z]/(x^2+y^2-z^2)$

not UFD.

$z \cdot z = (x+iy)(x-iy)$

Fixes: ① $k(V)$

② rational maps

Rational maps

$X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$

A rational map

$\varphi : X \dashrightarrow Y$

is an eq class of expressions

$[f_0 : \cdots : f_m]$ s.t.

① $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ homog. of same deg.

② $[f_0(p) : \cdots : f_m(p)] \neq [0 : \cdots : 0]$ some $p \in X$

③ $\forall\, p \in X$ if $[f_0(p) : \cdots : f_m(p)]$ is defined, it is in Y.

Two expressions are equiv. if they are equal where both defined.

example: φ, ψ from start of class:

$$[x:y:z] \longmapsto \begin{cases} [x:y] & \text{on } U_x \\ [y:z] & \text{on } U_z \end{cases}$$

Q. Why transitive?

Say φ is <u>regular</u> at p if $\varphi(p)$ defined for some expression representing φ.

So φ is <u>not</u> defined at non-reg. pts.

Rat. maps are like morphisms, but only def on open subset of X.

Example Cremona transformation.

$\varphi : \mathbb{P}^2 \to \mathbb{P}^2$

$[x:y:z] \mapsto [yz : xz : xy]$

Not def. at $[0:0:1]$ or any pt with two zeros.

In other words: $Z(x,y) \cup Z(y,z) \cup Z(x,z)$

Problem with rat'l maps: can't nec. compose $f \circ g$ if $g(\operatorname{dom} g) \cap \operatorname{dom} f = \emptyset$.

Dominant maps

$\varphi : X \dashrightarrow Y$ is **dominant** if $\varphi(\operatorname{dom} \varphi)$ nonempty open in Y

If φ dominant, can compose $\psi \circ \varphi$

Example Cremona map is dominant

and $\varphi \circ \varphi \approx \mathrm{id}$.

↑ equal where both def.

$[x:y:z] \overset{\varphi}{\mapsto} [yz : xz : xy]$

$\overset{\varphi}{\mapsto} [x^2yz : xy^2z : xyz^2]$

$= [x:y:z]$ if $x,y,z \neq 0$.

Field of rat'l fns

$k(X) = \{ f/g : f,g \in k[x_0,\dots,x_n]$ homog of same deg, $g \notin \mathbb{I}_p(X) \} / \sim$

$f_1/g_1 \sim f_2/g_2$ if $f_1g_2 - g_1f_2 \in \mathbb{I}_p(X)$

$\rightsquigarrow$ Well-def fns on X (open subset of X) $g \neq 0$.

Thm. ① A rat'l map $\varphi : X \dashrightarrow Y$ is birational (has rat'l inverse) $\iff$ φ dominant & φ^* is $\cong$

② X, Y birat. equiv $\longleftrightarrow$ $k(X) \cong k(Y)$

So: equiv of categories

$\{$ irred. quasi-proj vars $\}$ (open subset of var) $\longleftrightarrow$ $\{$ fg field exts of k $\}$

w/ birat. maps　　　　w/ k-homoms.