Algebraic Topology

Dan Margalit
Georgia Tech
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What is algebraic topology?

$$\boxed{\text{Space}} \longrightarrow \boxed{\text{Group}}$$

$X \rightleftarrows \pi_1(X)$ fundamental group

$X \longrightarrow H_k(X)$ k-th homology group

$X \longrightarrow H^k(X)$ k-th cohomology group

What kinds of questions does it answer?

① When are two spaces the same (or not)?

e.g. $\mathbb{R}^m \not\cong \mathbb{R}^n$

what about: $\mathbb{R}^3 -$  vs. $\mathbb{R}^3 -$ 

② Embeddings

What is smallest N s.t. a given manifold embeds in $\mathbb{R}^N$?

Unsolved for $\mathbb{RP}^n$.

③ Fixed point theorems

Brouwer fixed pt theorem: every $D^2 \to D^2$ has a fixed pt.

Borsuk-Ulam theorem.

④ Actions

Which finite groups act freely on S^n?
(known in some cases)
Note: $\mathbb{Z}/n\mathbb{Z} \circlearrowright S^{2k-1}$ $\forall n, k$.

⑤ Sections

What is the largest k s.t. a given manifold admits a continuously varying k-plane field?

Hairy ball theorem.

⑥ Group theory

Every subgroup of a free group is free.
$[F_n, F_n]$ is not finitely generated.
Braid groups are torsion free.

⑦ Algebra

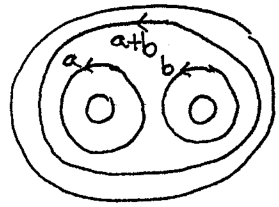
Fundamental theorem of algebra (this week!)

<u>Basic idea of homology</u>

$H_k(X)$ = abelian group of k-dim holes in X
↳ prevents a k-sphere from collapsing

<u>computable</u>

example: X = pair of pants
$H_1(X) \cong \mathbb{Z}^2$



$H^k(X)$ is dual to $H_k(X)$
$\rightsquigarrow$ consists of functions $H_k(X) \to \mathbb{Z}$

Big Goal: Poincaré Duality

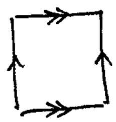
For X = n-manifold $H^k(X) \cong H_{n-k}(X)$

More precisely: the functions in H^k look like "intersect with this fixed element of H_{n-k}"

What do we mean by a space?

C = closure finiteness
(closure of open cell hits finitely many open cells)
W = weak topology

Cell complexes aka CW complexes

e.g. 

Quotient topology: $U \subseteq X/\sim$ is open iff its preimage in X is open.

We build CW complexes inductively

(i) Start with a discrete set of points X^0.
The points are regarded as 0-cells.

(ii) Inductively form <u>n-skeleton</u> X^n from X^{n-1} by attaching n-cells D^n_α via
$$\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$$

X^n has quotient topology.

(iii) Either stop at a finite stage, or continue indefinitely.

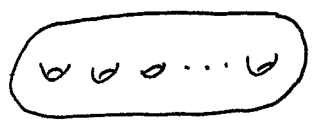
In latter case, use weak topology: a set is open iff its intersection with each cell is open.

$\dim(X) = \sup$ of dim of cells

Examples of CW Complexes

① 1-dim CW complexes are graphs.

② $(4g+2)$-gon with opposite sides identified



③ $S^n = e^0 \cup e^n$ $\qquad$ $e^i = i$-cell.

④ $\mathbb{RP}^n$ = space of lines in $\mathbb{R}^{n+1}$
$= e^0 \cup e^1 \cup \cdots \cup e^n$

To see this: $\mathbb{RP}^n = \cancel{\mathbb{R}^{n+1}}\ S^n /$ antipodal map
$= D^n /$ antipodal map on $\partial D^n = S^{n-1}$
So on ∂D^n see $\mathbb{RP}^{n-1}$, and we glue D^n to that.

⑤ $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^n$ $\qquad$ exercise.

Subcomplexes

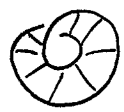
subcomplex = closed ~~subsp~~ union of cells.

A subcomplex of a CW complex is a CW complex.

example: k-skeleton.

Equivalence of Spaces

Intuition: Two spaces are equivalent if one can be deformed into the other

 $\sim$ A  $\sim$ O

Special case: A <u>deformation</u> <u>retraction</u> $X \to A$ is a continuous family

$$\{ f_t : X \to X \mid t \in I \}$$

s.t.
$$f_0 = id$$
$$f_1(X) = A$$
$$f_t|_A = id \quad \forall\, t.$$

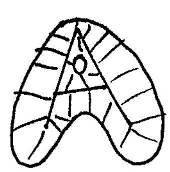
Continuous means
$$X \times I \to X$$
$$(x,t) \mapsto f_t(x)$$

is continuous.

Example: Given $f: X \to Y$, the <u>mapping</u> <u>cylinder</u> is

$$M_f = (X \times I) \amalg Y \,/\sim$$

where $(x,1) \sim f(x)$

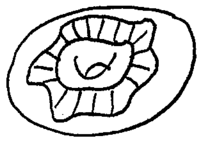
e.g.  X = boundary
Y = core

Fact: M_f deformation retracts to Y.

Homotopy Equivalence

A <u>homotopy</u> is a continuous family
$$\{ f_t : X \to Y \mid t \in I \}$$

examples: deformation retraction



A map $f: X \to Y$ is a <u>homotopy equivalence</u>
if there is a $g: Y \to X$ such that
$$fg \simeq id \text{ and } gf \simeq id$$
↑ homotopic

Say: X & Y are homotopy equivalent, or have the same homotopy type. $X \simeq Y$

Exercise: This is an equivalence relation.

Fact: If A is a deformation retract of X, then $X \simeq A$

Exercise:  all homotopy equiv.

Exercise: $\mathbb{R}^n \simeq *$ Say $\mathbb{R}^n$ is contractible.

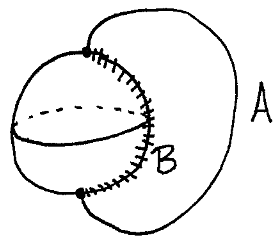
Read: House with 2 rooms, Hatcher p. 4.

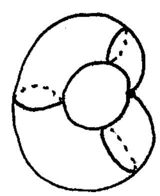
Two Criteria for Homotopy Equivalence

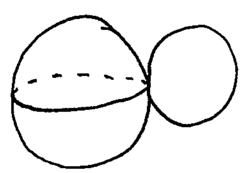
① (X, A) = CW-pair (i.e. A subcomplex of X)
A contractible
$\Longrightarrow X \simeq X/A$ ← identify A to one point

example: X = graph
A = any edge connecting distinct vertices.

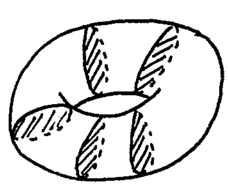
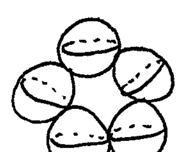
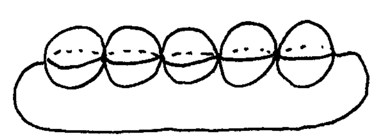
Thus any graph $\simeq$  = wedge of circles $S^1 \vee \cdots \vee S^1$

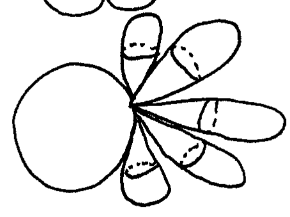
example: $X =$ 

$X/A \cong$  $\cong S^2 / $ north pole ~ south pole

$X/B \cong$  $= S^2 \vee S^1$

$X \simeq X/A \simeq X/B$

exercise:  $\simeq$  $\cong$ 

$\simeq$ 

② (X,A) CW-pair
$f,g : A \to Y$ homotopic (i.e. $\exists$ homotopy f_t, $f_0=f, f_1=g$)
$\Longrightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

Note: $X \sqcup_f Y = (X \amalg Y)/a \sim f(a)$

exercise: Do last example using Criterion ②

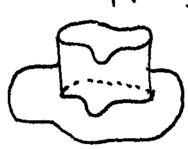
Proofs of both criteria use Homotopy Extension Property.

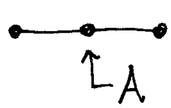
Say a pair of spaces (X,A) has the homotopy extension property if whenever we have

$$f_0 : X \to Y$$
$$f_t : A \to Y \quad \text{homotopy}$$

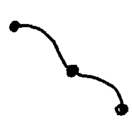
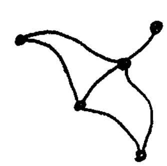
we can extend f_t to X.

In other words every map $M_i \to Y$ can be extended to $X \times I \to Y$
where M_i = mapping ~~cyl~~ cylinder of $i : A \to X$ inclusion.



example. $X =$ 

$Y = \mathbb{R}^2$

$f_0 =$  $f_t =$  extension: 

A <u>retraction</u> of a space X onto a subspace A is

$$r : X \to A$$

s.t. $r|_A = \mathrm{id}$.

<u>Prop</u>: (X,A) has HEP $\iff$ M_i is a retract of $X \times I$ where $i : A \to X$ inclusion.

<u>Proof</u>: $\Rightarrow$ Set $Y = M_i$, $f_0 = \mathrm{id}$.

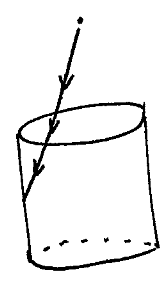
$\Leftarrow$ $X \times I \xrightarrow{r} M_i \xrightarrow{f_t} Y$

Note: f_t deformation retract of X to A
$\Rightarrow f_1 : X \to A$ a retraction of X to A

<u>Prop</u>: If $(X,A) =$ CW pair, then M_i is a deformation retract of $X \times I$ (where $i : A \to X$ incl.)

In particular, (X,A) has HEP.

<u>Proof</u>: First do $X = D^n$ $A = \partial D^n$ via projection:



Retract each n-cell of $X^n - A^n$ during $[\frac{1}{2}^{n+1}, \frac{1}{2}^n]$

Continuous since it is on each cell (no problem near 0 since each n-skeleton stationary in $[0, \frac{1}{2}^{n+1}]$).

Prop: (X,A) has HEP
A contractible
$\Rightarrow q: X \to X/A$ is a homotopy equivalence

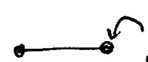
Idea: Need inverse to q. Contract A, extend to $f_t : X \to X$.
Since $f_1(A) = pt.$ can regard $f_1 : X/A \to X$.

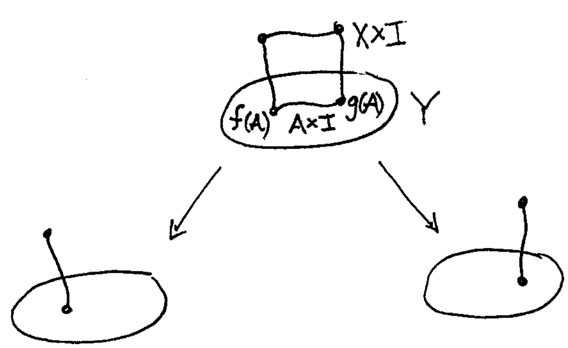
exercise: read/write details.

example. $X = \mathbb{R}$ $A = [-1,1]$

Prop: $(X,A) = CW$ pair
$f,g : A \to Y$ homotopic
$\Rightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

Idea: Show both are deformation retractions of
$$(X \times I) \sqcup_F Y$$
where $F: A \times I \to Y$ is homotopy from f to g.

example: $X =$  A $Y = D^2$



exercise: write details

note: use existence of deformation retraction $X \times I \to M_i$
(stronger than HEP).

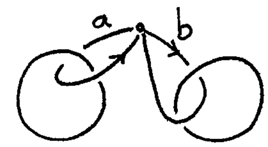
FUNDAMENTAL GROUP

$\pi_1(X)$ = group of homotopy classes of based paths in X.

Will see: $X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$

Examples: ① $\mathbb{R}^3$ - unknot $\rightsquigarrow \mathbb{Z}$

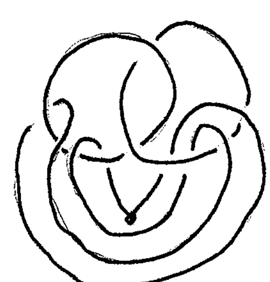
② $\mathbb{R}^3$ - unlink



$aba^{-1}b^{-1}$:

③ $\mathbb{R}^3$ - Hopf link

$aba^{-1}b^{-1}$:



= id

Is π_1 abelian?

push these two strands in tandem around the left-hand circle to see triviality.

Formal Definitions

A <u>path</u> in a space X is a map $I \to X$

A <u>homotopy of paths</u> is a homotopy $f_t : I \to X$ such that $f_t(0)$ and $f_t(1)$ are independent of t.

example. Any two paths f_0, f_1 in $\mathbb{R}^n$ with same endpoints are homotopic via straight-line homotopy:

$$f_t(s) = (1-t)\, f_0(s) + t f_1(s)$$

exercise. Homotopy of paths is an equivalence relation. $\simeq$

The <u>composition</u> of paths f, g with $f(1) = g(0)$ is the path

$$fg(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

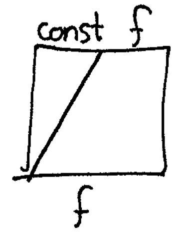
exercise. $f_0 \simeq f_1$, $g_0 \simeq g_1 \Rightarrow f_0 g_0 \simeq f_1 g_1$

A <u>loop</u> is a path f with $f(0) = f(1)$.

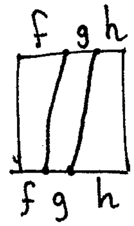
The fundamental group of X (based at x_0) is the group of homotopy classes of loops based at x_0 under composition. Write $\pi_1(X, x_0)$.

Prop: $\pi_1(X, x_0)$ is a group.

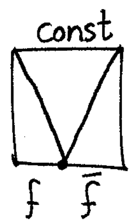
Proof: Identity = constant loop



Associativity:



Inverses:



$\bar{f}(t) = f(1-t)$

Prop: X = path connected, $x_0, x_1 \in X$
$\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1)$

The isomorphism is not canonical!

Say X is simply connected if ① X is path connected
② $\pi_1(X) = 1$.

This terminology is explained by:

Prop: X is simply connected $\Longleftrightarrow$ there is a unique homotopy class of paths joining any two points of X.

Fact: Contractible $\Rightarrow$ simply connected.

FUNDAMENTAL GROUP OF THE CIRCLE

<u>Thm</u>: $\pi_1(S^1) \cong \mathbb{Z}$

<u>Proof outline</u>: Given a loop $f : I \to S^1$, want to find a <u>lift</u>, that is:

$$\tilde{f} : I \to \mathbb{R}$$

such that $\tilde{f}(0) = 0$, $p\tilde{f} = f$ ← ignore the international date line.

The map $\pi_1(S^1) \to \mathbb{Z}$ is $f \mapsto \tilde{f}(1)$

Well-definedness: existence/uniqueness of lifts
Multiplicativity: easy
Injectivity: homotopic loops have homotopic lifts
Surjectivity: easy

Remains to show loops lift uniquely and homotopies lift.

Idea: Cover S^1 by small pieces whose preimages in $\mathbb{R}$ are unions of open intervals.
Given a loop/homotopy, cut it into pieces, lift piece by piece.

Proof thus follows from Lemma below.

Lemma: Given $F : Y \times I \to S^1$

$\tilde{F} : Y \times \{0\} \to \mathbb{R}$ lift of $F|_{Y \times \{0\}}$

$\exists!\ \tilde{F} : Y \times I \to \mathbb{R}$ lifting F, extending $\tilde{F}|_{Y \times \{0\}}$.

Path lifting: $Y = \{y_0\}$ Homotopy lifting: $Y = I$.

Proof ($Y = \{y_0\}$ case): Write I for $y_0 \times I$.

Cover S^1 by $\{U_\alpha\}$ so that $\forall \alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α.

F continuous, I compact $\Rightarrow$ can choose $0 = t_0 < t_1 < \cdots < t_m = 1$ so that $\forall i$, $F([t_i, t_{i+1}])$ is contained in some U_α; call it U_i.

Say $\tilde{F}$ defined on $[0, t_i]$, $\tilde{F}(t_i) \in \tilde{U}_i$, $p|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ homeo.

Define $\tilde{F}$ on $[t_i, t_{i+1}]$ via

$$(p|_{\tilde{U}_i})^{-1} \circ F|_{[t_i, t_{i+1}]}$$

Induct. ▨

Exercise. Prove for general Y.

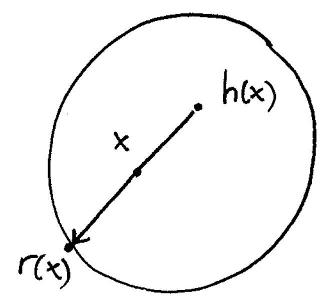
Prop: $\pi_1(X\times Y) \cong \pi_1(X) \times \pi_1(Y)$
for X, Y path connected.

Cor: $\pi_1(T^2) \cong \mathbb{Z}^2$

Applications

Brouwer Fixed Point Theorem: *Every $h: D^2 \to D^2$ has a fixed point.*

Proof: Say $h(x) \neq x \ \forall \ x \in D^2$.
Can define $r: D^2 \to S^1$ via
retraction



Let f_0 = loop in $S^1 = \partial D^2$
f_t = any homotopy to a point in D^2
$\Rightarrow r f_t$ = homotopy in S^1 of f_0 to trivial loop.
Thus $\pi_1(S^1) = 1$. Contradiction ▨

Also:

Borsuk-Ulam theorem – for any $f: S^2 \to \mathbb{R}^2$, $\exists$ antipodal pair $x, -x$ s.t. $f(x) = f(-x)$.

Ham Sandwich theorem.

Thm: If we write S^2 as a union of 3 closed sets, at least one must contain a pair of antipodal points.

Fundamental Theorem of Algebra: *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

<u>Proof</u>: Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$

Define $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$,

$\pi : \mathbb{C} - 0 \to S^1$

$\alpha \mapsto \alpha/|\alpha|$,

$R > |a_1| + \cdots + |a_n| + 1$,

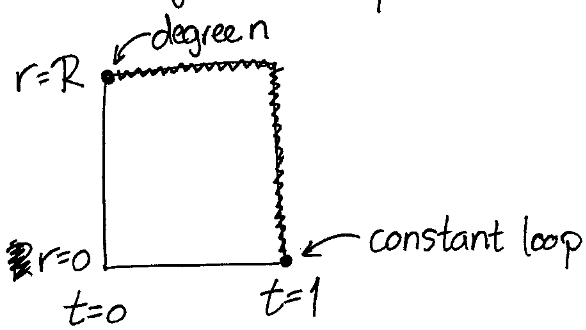
$f_{r,t}(s) : S^1 \to S^1$

$f_{r,t}(s) = \pi \circ p_t(re^{2\pi i s})$

<u>Claim</u>: p_t has no roots on $|z| = R$ for $t \in I$.

$\Rightarrow f_{R,t}(s)$ defined.

Thus the shaded path gives a homotopy from constant loop in S^1 to degree n loop $\Rightarrow n = 0$.



Proof of Claim: For $|z| = R$,

$$|z^n| = R^n = R \cdot R^{n-1} > (|a_1| + \cdots + |a_n|)\,|z^{n-1}|$$

$$\geq |a_1 z^{n-1} + \cdots + a_n|$$

(But $|\alpha| > |\beta| \Rightarrow \alpha + \beta \neq 0$.) ▨

Induced Homomorphisms

$$\varphi : (X, x_0) \longrightarrow (Y, y_0)$$
$$\rightsquigarrow \varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[f] \longmapsto [\varphi f]$$

Functoriality ① $(\varphi\psi)_* = \varphi_* \psi_*$
② $id_* = id$

<u>Fact</u>: φ a homeomorphism $\Rightarrow$ φ_* an isomorphism
<u>Proof</u>: $\varphi_* \varphi_*^{-1} = (\varphi\varphi^{-1})_* = id_* = id$

<u>Prop</u>: $\pi_1(S^n) = 1$ for $n \geq 2$.
<u>Proof</u>: $S^n - pt \cong \mathbb{R}^n$, which is contractible.
By Fact, suffices to show any loop in S^n is homotopic to one that is not surjective.

<u>Prop</u>: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, $n > 2$.
<u>Proof</u>: $\mathbb{R}^n - pt \cong S^{n-1} \times \mathbb{R}$

$$\pi_1(S^{n-1} \times \mathbb{R}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$$
$$\cong \begin{cases} \mathbb{Z} & n=2 \\ 1 & n>2 \end{cases}$$

Apply Fact.

Prop: If $\varphi : X \to Y$ homotopy equivalence, then
$\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ isomorphism.

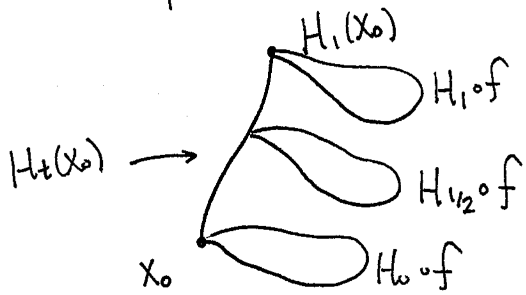
Proof: Let $\psi : Y \to X$ homotopy inverse.
So $\varphi\psi \simeq id$.

Remains to show: $H_t : X \to X$ homotopy
$H_0 = id$
$\Rightarrow (H_1)_* : \pi_1(X, x_0) \to \pi_1(X, H_1(x_0))$
an isomorphism.

We already know the path $H_t(x_0)$ gives

$$\pi_1(X, x_0) \xrightarrow{\cong} \pi_1(X, H_1(x_0))$$
$$f \longmapsto \overline{H_t(x_0)}\, f\, H_t(x_0)$$

But latter path $\simeq H_1 \circ f = (H_1)_*(f)$



So $(H_1)_*$ an isomorphism. ∎

Prop: $i: A \to X$ inclusion.
X retracts to $A \Rightarrow i_*$ injective
X deformation retracts to $A \Rightarrow i_*$ isomorphism.

exercise. T^2 retracts to S^1.

In group theory, a <u>retraction</u> is a homomorphism
$\rho: G \to H$, where $H < G$, with $\rho|_H = id$.
$\Rightarrow G \cong H \rtimes \ker \rho$.

FREE GROUPS AND FREE PRODUCTS

$F_n = \{$reduced words in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$
multiplication: concatenate, reduce.   associativity nontrivial!

$G * H = \{$reduced words in $G, H\}$
$*_\alpha G_\alpha$ similar $= \{g_1 \cdots g_m \quad g_i \in G_{\alpha_i} \quad \alpha_i \neq \alpha_{i+1} \quad g_i \neq id\}$

example. Infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$
= symmetries of $\cdots$

Properties
① $G_\alpha \leq * G_\alpha$
② $\bigcap G_\alpha = 1$
③ Any collection $G_\alpha \to H$
extends uniquely to $* G_\alpha \to H$

Van Kampen's Theorem

$X = A \cup B$ A, B open, path connected.
$A \cap B$ path connected.

$x_0 \in A \cap B$ basepoint for $X, A, B, A\cap B$.

The induced $\pi_1(A) \longrightarrow \pi_1(X)$ & $\pi_1(B) \longrightarrow \pi_1(X)$ extend to

$$\Phi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$$

Denote $i_A : A \to X$, $i_B : B \to X$.

Let N = normal subgroup of $\pi_1(A) * \pi_1(B)$ generated by the $i_A(\omega)\, i_B(\omega)^{-1}$ for $\omega \in \pi_1(A\cap B)$.

<u>Theorem</u>: ① Φ is surjective
② $\ker \Phi = N$.

Examples. ① $\pi_1(S^1 \vee S^1) \cong F_2$

induction $\rightsquigarrow$ $\pi_1(\bigvee_n S^1) \cong F_n$

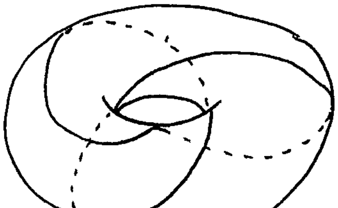
$\Rightarrow \pi_1(\mathbb{R}^2 - n \text{ pts}) \cong \pi_1(\mathbb{R}^3 - \text{unlink}) \cong F_n$.

$\pi_1(\text{graph}) \cong F_n$.

② $\pi_1(S^n) = 1 \quad n \geq 2$.

③ $\pi_1(S^3 - (p,q)\text{-torus knot}) \cong \langle x, y \mid x^p = y^q \rangle$

gluing two solid tori along an annulus.



VAN KAMPEN VIA PRESENTATIONS.

$$G_1 \cong \langle S_1 \mid R_1 \rangle$$
$$G_2 \cong \langle S_2 \mid R_2 \rangle$$
$$\Rightarrow G_1 * G_2 \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

What is a presentation for $\pi_1(A) * \pi_1(B) / N$?

First, a given $\gamma \in \pi_1(A \cap B)$ gives two elements of $\pi_1(A) * \pi_1(B)$:

$$\begin{tikzcd} & \pi_1(A) \arrow[dr] & \\ \pi_1(A \cap B) \arrow[ur] \arrow[dr] & & \pi_1(A) * \pi_1(B) \\ & \pi_1(B) \arrow[ur] & \end{tikzcd}$$

call them γ_A & γ_B.

Choose a generating set S for $\pi_1(A \cap B)$.
Choose presentations:

$$\pi_1(A) \cong \langle S_1 \mid R_1 \rangle$$
$$\pi_1(B) \cong \langle S_2 \mid R_2 \rangle$$

so each S_i contains each γ_A or γ_B for $\gamma \in S$.

Then:

$$\pi_1(A) * \pi_1(B) / N \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R \rangle$$

where R is the set of relations
$$\gamma_A = \gamma_B$$
for $\gamma \in S$.

Proof ① Let $f: I \to X$ loop at x_0.
Choose $0 = s_0 < s_1 < \cdots < s_m = 1$
s.t. $f|_{[s_i, s_{i+1}]}$ is a path in either A or B;
call it f_i.
$\forall i$, choose path g_i in $A \cap B$ from x_0 to $f(s_i)$
The loop
$$(f_1 \bar{g}_1)(g_1 f_2 \bar{g}_2) \cdots (g_{m-1} f_m)$$
is homotopic to f, and is a composition of loops, each in A or B. $\Rightarrow f \in \operatorname{Im} \Phi$.

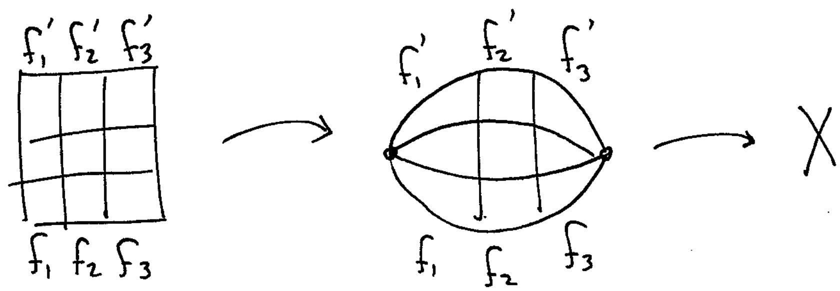
② A factorization of $f \in \pi_1(X)$ is an element of $\Phi^{-1}(f)$:
$$f_1 \cdots f_m \qquad f_i \in \pi_1(A) \text{ or } \pi_1(B)$$
We showed in ① that each f has a factorization.

Two factorizations are equivalent modulo N iff they differ by a sequence of moves:
(i) Combine $[f_i][f_{i+1}] \rightsquigarrow [f_i f_{i+1}]$
if f_i, f_{i+1} lie both in $\pi_1(A)$ or in $\pi_1(B)$.
(ii) Regard $[f_i] \in \pi_1(A)$ as $[f_i] \in \pi_1(B)$
if $f_i \in \pi_1(A \cap B)$.

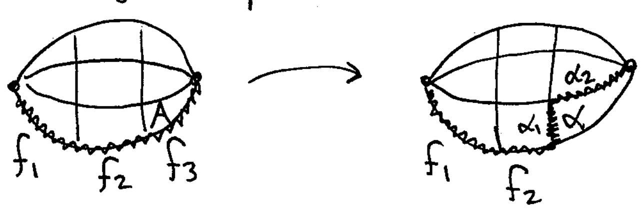
Let $f_1 \cdots f_k$, $f_1' \cdots f_\ell'$ factorizations of f.
To show they are related by (i) & (ii).

Choose a homotopy $I \times I \to X$ from one to the other.

Cut $I \times I$ into small rectangles, each mapping to A or B, and so induced partitions of top & bottom edges are finer than those coming from the factorizations.



Push across one square at a time. Show the new factorization differs from old by (i) & (ii).
E.g. two bottom-right squares.



$\alpha = f_3$ in $\pi_1(A)$

Then rewrite α as $\alpha_1 \alpha_2$ (move (i)).
rewrite α_1 as $\beta_1 \in \pi_1(B)$ (move (ii)).
Homotope $f_2 \beta_1 \in \pi_1(B)$ across square. etc. ▨

Attaching Disks

X path connected, based at x_0.
Attach 2-cell D^2 via $\varphi : S^1 \to X$.
$\rightsquigarrow Y$.
Choose path γ from x_0 to $\varphi(S^1)$.
The loop $\gamma\,\varphi(S^1)\,\bar{\gamma}$ is nullhomotopic in Y.
Let N = normal subgroup of $\pi_1(X)$ generated by this loop. Note: N independent of γ.

<u>Prop</u>. The inclusion $X \to Y$ induces a surjection
$$\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$$
with kernel N.

<u>Proof</u>: Choose $y \in \mathrm{int}(D^2)$
Apply Van Kampen to $Y - y$, $Y - X$.
Note: $Y - y \simeq X$
$Y - X \simeq *$
$(Y - y) \cap (Y - X) = \mathrm{int}(D^2) - y \simeq S^1$. ▨

<u>Applications</u>. ① M_g = orientable surface of genus g.
$\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$
$\Rightarrow M_g \not\simeq M_h$ $g \neq h$ as
$\pi_1(M_g)^{ab} \cong \mathbb{Z}^{2g}$.

(2) For any group G, there is a 2-dim cell complex X_G with $\pi_1(X_G) \cong G$.

To do this, choose a presentation

$$G = \langle g_\alpha \mid r_\beta \rangle$$

$X_G = \bigvee_\alpha S^1$ with 2-cells attached along r_β.

Covering Spaces.

In our proof of $\pi_1(S^1) \cong \mathbb{Z}$ we used $\mathbb{R} \to S^1$.
Can similarly show $\pi_1(T^2) \cong \mathbb{Z}^2$ using $\mathbb{R}^2 \to T^2$
or $\pi_1(S^1 \vee S^1) \cong F_2$ using $T_4 \to S^1 \vee S^1$.
In each case, $\pi_1(X)$ gives symmetries of the space lying above.

A <u>covering space</u> of X is an $\tilde{X}$ connected. with
$$p: \tilde{X} \to X$$
satisfying: $\exists$ open cover $\{U_\alpha\}$ of X so that each $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α.

Examples. $\mathbb{R} \to S^1$ $\quad \mathbb{R} \times I \to S^1 \times I$ $\quad \mathbb{R}^2 \to T^2$ $\quad S^2 \to \mathbb{RP}^2$

$S^1 \xrightarrow{\times n} S^1$ $\quad \mathbb{R} \times I \to$ Möbius Strip $\quad \mathbb{R}^2 \to$ Klein bottle

A <u>universal</u> covering space is a covering space that is simply connected.

We will see: ① $\pi_1(X) \leftrightarrow$ symmetries of univ. cover $\tilde{X}$

② Subgroups of $\pi_1(X) \leftrightarrow$ covers of X.

e.g. $X = S^1$.

① via path lifting, ② via path projecting

FUNDAMENTAL THEOREM

$p : \tilde{X} \to X$ covering map

$G(\tilde{X})$ = deck transformation group
= p-equivariant symmetries of $\tilde{X}$:

$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[dl] \\ & X & \end{tikzcd}$$

$H = p_* \pi_1(\tilde{X})$, $N(H)$ = normalizer in $\pi_1(X)$.

Theorem $1 \to H \to N(H) \to G(\tilde{X}) \to 1$

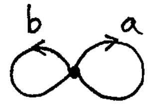
The map $N(H) \longrightarrow G(\tilde{X})$ is
$\gamma \longmapsto$ unique deck trans taking $\tilde{x}_0$ to $\tilde{\gamma}(1)$.

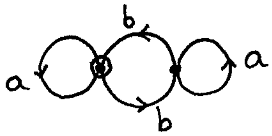
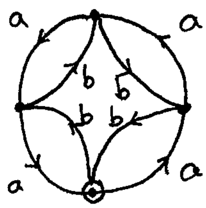
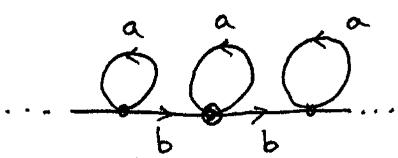
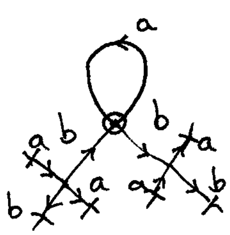
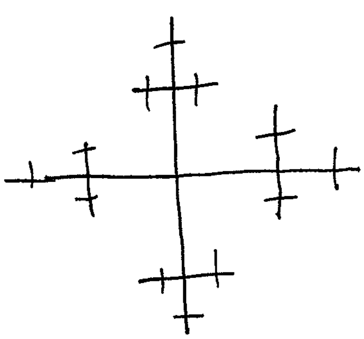
Cor: $H = 1 \Leftrightarrow G(\tilde{X}) \cong \pi_1(X) \Leftrightarrow \tilde{X}$ = universal cover.

Cor: H normal $\Leftrightarrow G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$.

There is a bijection:

$$\left\{\begin{array}{c}\text{based covering} \\ \text{spaces of } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subgroups} \\ \text{of } \pi_1(X)\end{array}\right\}$$

EXAMPLE $X =$ 

$\tilde{X}$	$p_*(\pi_1(\tilde{X}))$
	$\langle a, b^2, bab^{-1} \rangle$
	$\langle a^2, b^2, ab \rangle$
	$\langle a^4, b^4, ab, ba, a^2b^2 \rangle$
	$\langle b^n a b^{-n} \rangle$
	$\langle a \rangle$
	1

Lifting Properties

$p: \tilde{X} \to X$ covering space

A <u>lift</u> of $f: Y \to X$ is $\tilde{f}: Y \to \tilde{X}$ with $p\tilde{f} = f$.

Proposition 1 (Homotopy lifting property) Given a homotopy $f_t: Y \to X$ and $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0, $\exists!\ \tilde{f}_t$ lifting f_t.

Proof: Same as S^1 case.

$Y =$ point $\rightsquigarrow$ path lifting property
$Y = I$ $\rightsquigarrow$ homotopy lifting for paths

<u>Cor</u>: $p_*: \pi_1(\tilde{X}) \to \pi_1(X)$ is injective.

Note: $p_*(\pi_1(\tilde{X}))$ is the subgroup of $\pi_1(X)$ consisting of loops that lift to loops.

Degree of a cover: $|p^{-1}(x)|$ is locally constant, hence constant

<u>Cor</u>: $X, \tilde{X}$ path connected.

$$\text{degree of } p = [\pi_1(X) : p_*(\pi_1(\tilde{X}))]$$

<u>Proof</u>: Let $H = p_*\pi_1(\tilde{X})$.

Define $\{\text{cosets of } H\} \longrightarrow p^{-1}(x_0)$

$$H[g] \longmapsto \tilde{g}(1).$$

Surjective: path proj. Injective: path lifting ▨

Proposition 2 (Lifting existence criterion) Y = connected, locally path connected. We can lift $f : (Y, y_0) \to (X, x_0)$ to $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ iff

$$f_*(\pi_1(Y)) \leq p_* \pi_1(\tilde{X}).$$

<u>Proof</u>: ($\Rightarrow$) $\tilde{f}$ exists $\Rightarrow$ $f = p\tilde{f} \Rightarrow f_* = p_* \tilde{f}_*$

$$\Rightarrow \operatorname{Im} f_* \subseteq \operatorname{Im} p_*.$$

($\Leftarrow$) Suppose $\operatorname{Im} f_* \subseteq \operatorname{Im} p_*$. Want to <u>build</u> $\tilde{f}$.

Let $y \in Y$, γ a path from y_0 to y.
Prop 1 $\Rightarrow f\gamma$ has unique lift $\widetilde{f\gamma} : Y \to \tilde{X}$.
Define

$$\tilde{f}(y) = \widetilde{f\gamma}(1).$$

Why is $\tilde{f}$ well-defined?

Let γ' = another path from y_0 to y.

$\Rightarrow (f\gamma')(\overline{f\gamma})$ is a loop h_0 at x_0.

$\Rightarrow h_0 = f(\gamma\bar{\gamma}) \in f_*(\pi_1(Y))$

$\Rightarrow h_0 \in p_*(\pi_1(\tilde{X}))$ by assumption.

$\Rightarrow$ the lifted path $\tilde{h}_0$ is a <u>loop</u>.

Uniqueness of lifted paths $\Rightarrow \tilde{h}_0 = \widetilde{f\gamma}\,\overline{\widetilde{f\gamma'}}$

$\Rightarrow \widetilde{f\gamma}, \widetilde{f\gamma'}$ share common endpoint.

Exercise: $\tilde{f}$ continuous. ∎

Proposition 3 (Uniqueness of lifts) Let $f : Y \to X$, Y connected. If lifts $\tilde{f}_1$, $\tilde{f}_2$ agree at one point, then they are equal.

<u>Proof</u>: Will show

$$A = \{ y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y) \}$$

is open and closed in Y.

Let $y \in Y$. Let U be open nbhd of Y as in definition of covering space.

Let $\tilde{U}_1$, $\tilde{U}_2$ be the components of $p^{-1}(U)$ containing $\tilde{f}_1(y)$, $\tilde{f}_2(y)$.

Continuity of $\tilde{f}_i \Rightarrow \exists$ nbhd N of y with

$$\tilde{f}_i(N) \subseteq \tilde{U}_i$$

- $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \tilde{U}_1 \neq \tilde{U}_2 \Rightarrow \tilde{f}_1(N) \cap \tilde{f}_2(N) = \emptyset$
  $\Rightarrow A$ closed.
- $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{U}_1 = \tilde{U}_2 \Rightarrow \tilde{f}_1|_N = \tilde{f}_2|_N$
  Thus A open. ∎

Classification of Covering Spaces

$$\{\text{based covers of } X\} \leftrightarrow \{\text{subgroups of } \pi_1(X)\}$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

First step: find a cover corresponding to trivial subgroup.

<u>Theorem</u>: X = CW-complex (or any path conn, locally path conn, semilocally simply conn.)
Then X has a universal cover $\tilde{X}$.

<u>Proof</u>: We construct $\tilde{X}$ directly.

Points in $\tilde{X}$ $\leftrightarrow$ homotopy classes of paths from $\tilde{x}_0$ (simple connectivity)
$\leftrightarrow$ homotopy classes of paths from x_0 (homotopy lifting)

So define:

$$\tilde{X} = \{ [\gamma] : \gamma \text{ a path in } X \text{ at } x_0 \}$$

$$p : \tilde{X} \to X$$
$$[\gamma] \longmapsto \gamma(1)$$

Topology on $\tilde{X}$

$\mathcal{U} = \{ U \subseteq X : U \text{ path conn.}, \pi_1(U) \to \pi_1(X) \text{ trivial} \}$

For $U \in \mathcal{U}$, γ with $\gamma(1) \in U$, define

$$U_{[\gamma]} = \{ [\gamma \cdot \eta] : \eta \text{ a path in } U, \ \eta(0) = \gamma(1) \}$$
$$= \text{open neighborhood of } [\gamma] \text{ in } \tilde{X}.$$

exercise: The $U_{[\gamma]}$ form a basis.

We now check the properties of a covering space.

- Continuity. $p^{-1}(U)$ is a union of $U_{[\gamma]}$

- Path connectivity. Let $[\gamma] \in \tilde{X}$.

  $$\gamma_t = \begin{cases} \gamma \text{ on } [0,t] \\ \text{const. on } [t,1] \end{cases}$$

  is a path from [const] to $[\gamma]$.

- Simple connectivity. p_* injective, so suffices to show

  $$p_* \pi_1(\tilde{X}) = 1.$$

  Let $\gamma \in \operatorname{Im} p_* \Rightarrow \gamma$ lifts to a loop.

  The lift of γ is $\{[\gamma_t]\}$

  loop $\Rightarrow [\gamma_1] = [\gamma_0]$

  or $[\gamma] = [\text{const}]$

  $\Rightarrow \gamma = 1$ in $\pi_1(X)$.

· Covering Space. Note: If $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$

Thus, for fixed $U \in \mathcal{U}$, the $U_{[\gamma]}$ partition $p^{-1}(U)$

$p: U_{[\gamma]} \longrightarrow U$ homeomorphism since it gives a bijection of open sets

$$V_{[\gamma]} \subseteq U_{[\gamma]} \longleftrightarrow V \subseteq U$$

for $V \in \mathcal{U}$. ▨

Theorem: For every $H \leq \pi_1(X)$ there is a (based) covering space

$$p: \tilde{X}_H \longrightarrow X$$

with $p_* \pi_1(\tilde{X}_H, \tilde{x}_0) = H$.

Proof: We realize $\tilde{X}_H$ as a quotient $\tilde{X}_H = \tilde{X}/\sim$:

$$[\gamma] \sim [\gamma'] \quad \text{if} \quad \gamma(1) = \gamma'(1)$$
$$\text{and} \quad [\gamma \cdot \bar{\gamma}'] \in H.$$

exercise: $\sim$ is an equivalence relation.

Check $\tilde{X}_H$ a covering space:

Say $[\gamma] \sim [\gamma']$ with $\gamma(1) = \gamma'(1) \in U \in \mathcal{U}$.

Then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$ for any path η in U.

$\Rightarrow U_{[\gamma]}$ identified with $U_{[\gamma']}$

Check $p_* \pi_1(\tilde{X}_H) = H$:

Let $\tilde{x}_0 = [\text{const}]$.

$\gamma \in \operatorname{Im} p_* \iff \{[\gamma_t]\}$ a loop in $\tilde{X}_H$

$\iff [\gamma_0] \sim [\gamma_1]$

i.e. $[\text{const}] \sim [\gamma]$

$\iff \gamma \in H$. ▨

To finish classification, need to show $\tilde{X}_H$ unique.

Def: Covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ with $p_1 = p_2 f$ (i.e. f preserves fibers).

Prop: Two path connected covering spaces $p_1 : (\tilde{X}_1, \tilde{x}_1) \to X$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \to X$ are isomorphic if and only if

$$\operatorname{Im}(p_1)_* = \operatorname{Im}(p_2)_*.$$

Proof: $\Rightarrow$ easy.

$\Leftarrow$ Lifting criterion $\rightsquigarrow$ lift p_1 to $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \tilde{p}_1 = p_1$

By symmetry $\rightsquigarrow$ $\tilde{p}_2$ with $p_1 \tilde{p}_2 = p_2$.

Note $\tilde{p}_1 \tilde{p}_2$ is a lift of p_2:

$$p_2 \tilde{p}_1 \tilde{p}_2 = p_1 \tilde{p}_2 = p_2$$

Unique lifting + $\tilde{p}_1 \tilde{p}_2 (\tilde{x}_2) = \tilde{x}_2 \Rightarrow \tilde{p}_1 \tilde{p}_2 = id$.

Symmetry: $\tilde{p}_2 \tilde{p}_1 = id$.

$\Rightarrow \tilde{p}_1$ a homeo. ▨

Cor: Every subgroup of a free group is free.

Some Examples of Covering Spaces

$$S^1 \times \mathbb{R} \longrightarrow T^2$$

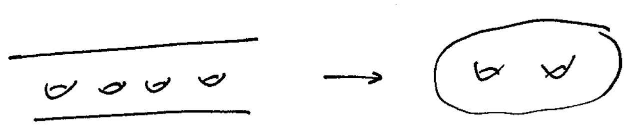
$$T^2 \xrightarrow{(\times m, \times n)} T^2$$

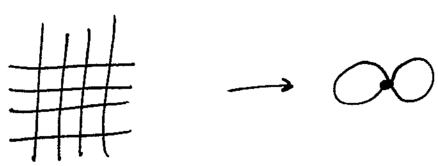
Annulus $\longrightarrow$ Möbius strip

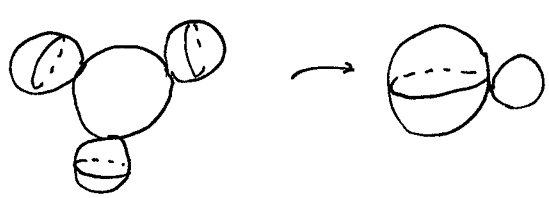
$$S^2 \longrightarrow \mathbb{RP}^2$$

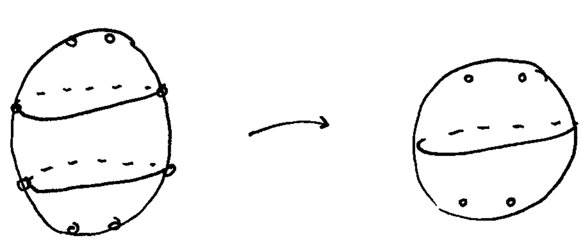
$$\mathbb{C}^* \xrightarrow{z^n} \mathbb{C}^*$$

$$\mathbb{C}^* \longrightarrow T^2$$









THE FUNDAMENTAL THEOREM

Fix $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$
$H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$
$N(H)$ = normalizer in $\pi_1(X, x_0)$
$G(\tilde{X})$ = group of deck transformations.

Say p is <u>regular</u> if $G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$.

Regard $\tilde{x}_0$ as [const]
Then $p^{-1}(x_0) = \{ [\gamma] : \gamma \text{ a loop} \}$
By lifting criterion,

$\exists$ deck trans I_γ taking [const] to $[\gamma]$

$\Leftrightarrow$ $p_* \pi_1(\tilde{X}, [\gamma]) = p_* \pi_1(\tilde{X}, [\text{const}])$

or $\gamma \; p_* \pi_1(\tilde{X}, [\text{const}]) \; \gamma^{-1} = p_* \pi_1(\tilde{X}, [\text{const}])$

i.e. $\gamma \in N(H)$.

We thus have:

$$\begin{aligned} N(H) &\longrightarrow G(\tilde{X}) \\ \gamma &\longmapsto I_\gamma \end{aligned}$$

Note: well-defined by uniqueness of lifts.

<u>Prop</u>: $\tilde{X}$ regular $\Leftrightarrow$ H normal.

<u>Theorem</u>: $G(\tilde{X}) \cong N(H)/H$

Both are exercises.

Covering Spaces via Actions

An <u>action</u> of a group G on a space Y is a homom:

$$G \longrightarrow \text{Homeo}(Y)$$

This is a <u>covering space action</u> if

$\forall\, y \in Y\ \exists$ neighborhood U with

$\{g(U) : g \in G\}$

all distinct, disjoint.

<u>Fact</u>: The action of $G(\tilde{X})$ on $\tilde{X}$ is a covering space action.

<u>Prop</u>: Y = connected CW-complex

(or any path conn, locally path conn)

$G \circlearrowleft Y$ via covering space action. Then:

(i) $p : Y \to Y/G$ a regular covering space.

(ii) $G \cong G(Y)$

In particular
- $G \cong \pi_1(Y/G) / p_* \pi_1(Y)$
- Y simply connected $\Rightarrow \pi_1(Y/G) \cong G$.

<u>Examples</u>.

$\mathbb{Z} \circlearrowleft \mathbb{R} \rightsquigarrow S^1$

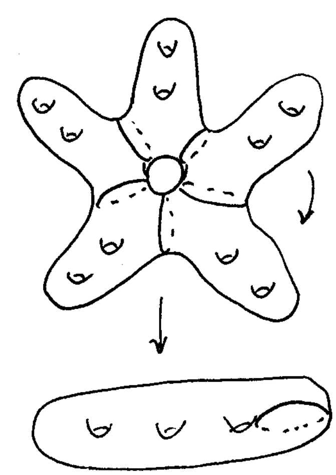
$\mathbb{Z} \circlearrowleft \mathbb{R} \times I \rightsquigarrow$ Möbius strip

$\mathbb{Z}^2 \circlearrowleft \mathbb{R}^2 \rightsquigarrow T^2$

Klein bottle

$\mathbb{Z}/2\mathbb{Z} \circlearrowleft S^n \rightsquigarrow \mathbb{RP}^n$

$\mathbb{Z}/m\mathbb{Z} \circlearrowleft M_{mk+1} \longrightarrow M_{k+1}$



K(G,1) Spaces

Goal: groups $\leftrightarrow$ spaces (up to homotopy equiv.)
homomorphisms $\leftrightarrow$ continuous maps (up to homotopy)

A $K(G,1)$ space is a space with fundamental group G and contractible universal cover.

Examples. S^1, T^2 in general $\mathbb{Z}^n \leftrightarrow T^n$

What about $G = \mathbb{Z}/m\mathbb{Z}$?

$\mathbb{Z}/m\mathbb{Z}$ acts on S^∞ = unit sphere in $\mathbb{C}^\infty$ via

$$(z_i) \mapsto e^{2\pi i m}(z_i)$$

which is a covering space action.
(When $m=2$, quotient is $\mathbb{R}P^\infty$).

Why is S^∞ contractible?

Step 1: $f_t(x_1, x_2, \ldots) = (1-t)(x_i) + t(0, x_1, x_2, \ldots)$

Step 2: Straight line projection to $(1,0,0,\ldots)$.

Later: Any $K(\mathbb{Z}/m\mathbb{Z},1)$ is ∞-dim!

CONSTRUCTION OF $K(G,1)$ spaces

<u>Prop</u>: Every group G has a $K(G,1)$

<u>Proof</u>: Define a Δ-complex EG with:

$$n\text{-simplices} \longleftrightarrow \begin{array}{l}\text{ordered}\\ (n+1)\text{-tuples}\\ [g_0,\dots,g_n] \quad g_i \in G\end{array}$$

To see EG contractible, slide each $x \in [g_0,\dots,g_n]$ along line segment in $[e,g_0,\dots,g_n]$ from x to $[e]$

(Note: This is not a deformation retraction since it moves $[e]$ around $[e,e]$.)

$G \circlearrowright EG$ by left multiplication.

exercise: This is a covering space action.

$\rightsquigarrow BG = EG/G$ is a $K(G,1)$.

This gives one $K(G,1)$, and it is always ∞-dim.
To study a group G, need a <u>good</u> $K(G,1)$,
e.g. $K(PB_n,1) = \mathbb{C}^n \setminus \Delta$.

Homomorphisms as Maps

<u>Prop</u>: X = connected CW-complex
$Y = K(G,1)$
Every homomorphism $\pi_1(X, x_0) \longrightarrow \underset{\pi_1(Y,y_0)}{\overset{\shortparallel}{G}}$ is induced by a map $(X, x_0) \longrightarrow (Y, y_0)$.
The map is unique up to homotopy fixing y_0.

This implies:

<u>Prop</u>: The homotopy type of a CW-complex $K(G,1)$ is uniquely determined by G.

<u>Proof of 1st Prop</u>: Assume first X has one 0-cell, x_0.

Let $\varphi: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$. Want $f: X \to Y$.

Step 0. $f(x_0) = y_0$

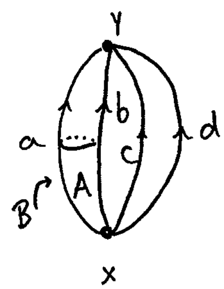
Step 1. Each edge e of X is an element of $\pi_1(X, x_0)$. Define $f(e)$ via φ.

Step 2. Let Δ = 2-cell with $\psi: \partial\Delta \longrightarrow X^{(1)}$
$f\psi$ null-homotopic, since φ a homom.
$\leadsto$ can extend f to Δ.

Homology

Fundamental groups are good at telling spaces apart, but it is not so easy to compute, and the higher dimensional analogs are very hard to compute. Indeed: computing $\pi_m(S^n)$ is a huge open problem.

Homology is an analogue that is computable. We will lose some information, but it will still be possible to tell many spaces apart.

Example. $X =$ 

$A =$ front of sphere
$B =$ back.

$C_0 =$ free abelian group on x, y
$C_1 =$ free abelian group on a, b, c, d
$C_2 =$ free abelian group on A, B.

An element of $H_1(X)$ is a 1-cycle: an element of C_1 with no boundary, e.g. ab^{-1}.

Since C_1 abelian, $ab^{-1} = b^{-1}a$ so we think of ab^{-1} as a loop with no basepoint.

A 1-cycle is trivial if it is the boundary of a 2-cell, or a collection of 2-cells, so:

ab^{-1} trivial, cd^{-1} not.

In other words, $H_1(X) = {}^{\text{1-cycles}}/_{\text{1-boundaries}}$.

Can compute with linear algebra.

$$\partial_1 : C_1 \to C_0 \qquad \text{"boundary map"}$$
$$a,b,c,d \mapsto y-x$$

$$\text{1-cycles} = \ker \partial_1.$$

$$\partial_2 : C_2 \to C_1$$
$$A,B \mapsto a-b$$

$$\text{1-boundaries} = \operatorname{im} \partial_2.$$

So: $H_1(X) = {}^{\ker \partial_1}/_{\operatorname{im} \partial_2}$

Exercise: $\ker \partial_1 = \langle a-b,\ b-c,\ c-d \rangle \cong \mathbb{Z}^3$

$\operatorname{im} \partial_2 = \langle a-b \rangle$

essentially lin. alg.

$\Rightarrow H_1(X) \cong \mathbb{Z}^2$

Also: $H_2(X) = {}^{\ker \partial_2}/_{\operatorname{im} \partial_3} = {}^{\langle A-B \rangle}/_{1} \cong \mathbb{Z}.$

Simplicial Homology

$X = \Delta$-complex

$\Delta_n(X) =$ free abelian group on n-simplices of X.

$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

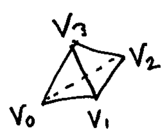
$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

There is also singular homology: $X =$ any space

$C_n(X) =$ free abelian group on all maps $\Delta^n \to X$.

More complicated, but more powerful. Will turn out to be equivalent.

Δ-complexes

Simplex 

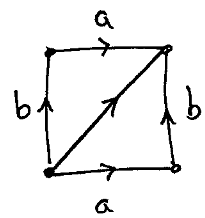
ordering of vertices $\rightsquigarrow$ ordering of vertices for each face.

To build a Δ-complex:

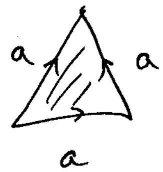
- Start with a discrete set of vertices
- Attach edges to produce a graph.
- Attach 2-simplices along edges, respecting orderings of vertices
- etc.

$\Delta_n(X) =$ free abelian group on n-simplices.

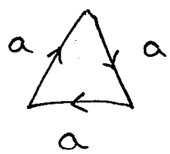
Exercise: every simplicial complex has the structure of a Δ-complex.

Examples. ① 

$\cong T^2$

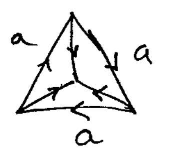
② 

Dunce hat.

③ 

Nonexample. No way to order vertices of 2-simplex consistently with edges.

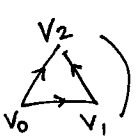
Here is a Δ-complex structure on same space:



Boundary homomorphism

$$\partial([v_0, \dots, v_n]) = \sum (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

e.g. $\partial([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

or: $\partial\Big($ 

$\Big) =$ 

where $[v_0, \dots, v_n]$ is Δ^n = standard n simplex.

For a simplex $\sigma : \Delta^n \to X$ in Δ-complex:

$$\partial\sigma(\Delta^n) = \sigma(\partial\Delta^n).$$

Lemma: $\partial_{n-1} \circ \partial_n = 0$.

Proof: Check on one simplex $\Delta = [v_0, \dots, v_n]$

$$\partial_n(\Delta) = \sum (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

$$\partial_{n-1}\partial_n(\Delta) = \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

$$+ \sum_{j>i} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= 0. \qquad \text{(switch roles of } i \text{ \& } j \text{ in last sum).}$$

We now have:

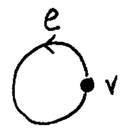
$$\cdots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \longrightarrow 0$$

with $\partial_n \partial_{n+1} = 0 \ \forall n$. i.e. $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$

This is called a chain complex.

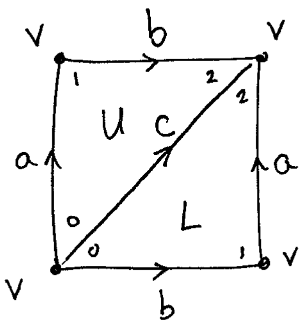
$\rightsquigarrow$ can define: $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1} = n\text{-cycles}/n\text{-boundaries}$

"n^{th} homology group of X"

EXAMPLES. ① $X = S^1 =$ 

$$\Delta_0(X) = \langle v \rangle \cong \mathbb{Z}$$
$$\Delta_1(X) = \langle e \rangle \cong \mathbb{Z}$$
$$\partial_1 = 0 \qquad \partial_1(e) = v - v = 0.$$
$$\rightsquigarrow \quad H_n(X) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

② $X = T^2$

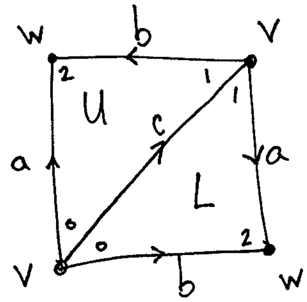


$$\partial_1 = 0 \qquad \partial_0 = \partial_3 = 0.$$
$$\partial_2(U) = \partial_2(L) = a + b - c$$

$$H_0(X) = \langle v \rangle / 0 \cong \mathbb{Z}$$

$$H_1(X) = \langle a, b, c \rangle / \langle a + b - c \rangle \cong \langle a, b \rangle \cong \mathbb{Z}^2$$
$$H_2(X) = \langle U - L \rangle / 0 \cong \mathbb{Z}.$$

③ $X = \mathbb{RP}^2$



$$H_0(X) = \langle v, w\rangle / \langle v - w\rangle = \mathbb{Z}$$

$$\ker \partial_1 = \langle a-b, c\rangle = \langle c, a-b+c\rangle \cong \mathbb{Z}^2$$
$$\operatorname{im} \partial_2 = \langle a+b+c, a-b+c\rangle = \langle a-b+c, 2c\rangle \cong \mathbb{Z}^2$$

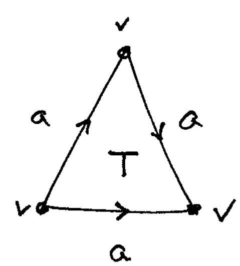
$$\rightsquigarrow H_1(X) = \langle c, a-b+c\rangle / \langle 2c, a-b+c\rangle \cong \mathbb{Z}/2\mathbb{Z}$$

Next: $\ker \partial_2$

$$\partial_2(pU + qL) = (q-p)a + (p-q)b + (p+q)c$$
$$\Longrightarrow \ker \partial_2 = 0.$$

$$\rightsquigarrow H_2(X) = 0.$$

④ X = Dunce cap

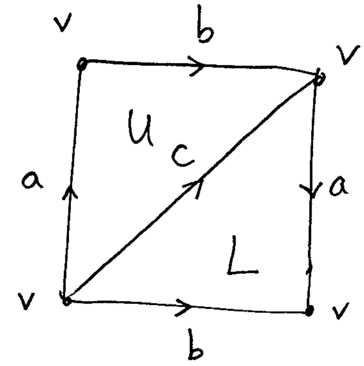


X is contractible
but not collapsible.

$$H_1(X) = \langle a\rangle / \langle a\rangle = 0$$
$$H_0(X) = \langle v\rangle / 0 \cong \mathbb{Z}$$
$$H_2(X) = 0$$

Exercise: $X \simeq *$ (it is mapping cone of deg 1 map $S^1 \to S^1$).

⑤ X = Klein bottle



$H_0(X) = \langle v \rangle / 0 \cong \mathbb{Z}$

$H_2(X) = 0.$

$\operatorname{Ker} \partial_1 = \langle a, b, c \rangle$

$\operatorname{Im} \partial_2 = \langle a+b-c,\ a-b+c \rangle$

How to compute quotient? Find Smith normal form of:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

i.e. use row/col ops to get diagonal matrix where each diagonal entry divides the next.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$H_1(X) \cong \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/0\mathbb{Z}$$
$$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Will prove: $H_1(X) \cong \pi_1(X)^{ab}$.

Singular Homology

Simplicial homology is very computable, but:

1. It is not obvious that homeomorphic Δ-complexes have isomorphic simplicial homology.
2. Hard to prove general facts about spaces.

So: A *singular n-simplex* in X is a map $\sigma : \Delta^n \to X$

Let $C_n(X)$ = free abelian group on these.

$$= \text{group of } n\text{-chains}$$

$$= \{ \sum n_i \sigma_i \mid n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \}$$

Boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\sigma \mapsto \sum (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Still have $\partial_{n-1} \circ \partial_n = 0$.

$$H_n(X) = {}^{\ker \partial_n} / {}_{\operatorname{Im} \partial_{n+1}}$$ "nth singular homology group"

Singular homology hard to compute. For example, not obvious that

1. $H_n(X) = 0$ for $n > \dim X$
2. $H_n(X)$ finitely gen.

On other hand, easy to prove general facts like:

Fact: Homeomorphic spaces have isomorphic singular hom. groups.

Will show: singular = simplicial.

Note. Elements of $H_1(X)$ rep. by maps $S^1 \to X$ (easy)

$H_2(X)$ rep. by maps $M_g \to X$ (less easy)

$H_n(X)$ rep by maps n-manifold $\to X$ (only true over $\mathbb{Q}$)

Prop: X = space with path components X_α
$\Rightarrow H_n(X) \cong \bigoplus H_n(X_\alpha)$

Prop: X = nonempty, path conn. $\Rightarrow H_0(X) \cong \mathbb{Z}$
X has n path comp. $\Rightarrow H_0(X) \cong \mathbb{Z}^n$

Proof: Say X path conn.

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$H_0(X) = C_0(X)/\operatorname{Im}\partial_1$

Given $v, w \in X$, $v - w \in \operatorname{Im}\partial_1 \Rightarrow v = w$ in $H_0(X)$.

Also, $nv \neq 0$ in $H_0(X)$ since $\operatorname{Im}\partial_1 \subseteq \ker\left(C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}\right)$

where $\varepsilon(\sum n_i v_i) = \sum n_i$. ∎

Prop: X = pt.

$$\Rightarrow H_i(X) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

Pf: $C_n(X) \cong \mathbb{Z} \ \forall n$.

$$\partial(\sigma_n) = \Sigma(-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$ ∎

Reduced Homology

Looking at last Prop, seems more elegant to replace last 0 map with $\cong$.

$$\tilde{H}_n(X) = \text{homology of } \cdots \to C_1(X) \longrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum n_i \sigma_i) = \sum n_i$

= reduced homology of X.

Exercise: $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$

In particular: $\tilde{H}_i(X) = 0 \ \forall i$ when X = pt.

Homotopy Invariance

Goal: $f : X \to Y \rightsquigarrow f_* : H_n(X) \to H_n(Y)$

and

f homotopy equivalence $\Rightarrow$ f_* an isomorphism.

First, $f \rightsquigarrow f_\# : C_n(X) \to C_n(Y)$

$$\sigma \mapsto f\sigma$$

with $f_\# \partial = \partial f_\# \rightsquigarrow$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

"chain map"

$f_\#$ takes cycles to cycles, boundaries to boundaries.

$\Rightarrow$ $f_\#$ induces $f_* : H_n(X) \to H_n(X)$

<u>Facts</u>: $(fg)_* = f_* g_*$

$id_* = id$

<u>Theorem</u>. $f, g : X \to Y$ homotopic $\Rightarrow f_* = g_*$

<u>Cor</u>: $f : X \to Y$ homotopy equiv. $\Rightarrow$ f_* an isomorphism.

example. X contractible $\Rightarrow \tilde{H}_i(X) = 0 \ \forall i$.

Proof of Theorem: We will define $P: C_n(X) \to C_{n+1}(Y)$ with

$$\partial P = g_\# - f_\# - P\partial \qquad \text{"prism operator"}$$

P is the homotopy from $f\sigma$ to $g\sigma$.

The theorem follows:

If $\alpha \in C_n(Y)$ is a cycle, then

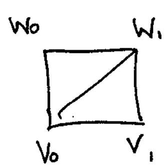
$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\underbrace{\partial(\alpha)}_{=0} = \partial P(\alpha)$$

$\Rightarrow (g_\# - f_\#)(\alpha)$ a boundary

$\Rightarrow$ ~~$g_\#(\alpha)$~~ $g_*(\alpha) = f_*(\alpha)$

Remains to define P and check $\partial P = g_\# - f_\# - P\partial$.

Main ingredient. Cutting $\Delta^n \times I$ into $(n+1)$-simplices

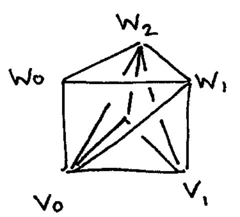


Label vertices of $\Delta^n \times 0$ by $v_0, \dots, v_n$

$\Delta^n \times 1$ by $w_0, \dots, w_n$.

$\Delta^n \times I$ decomposes as sum of

$[v_0, \dots, v_i, w_i, \dots, w_n]$



Define $P(\sigma) = \sum (-1)^i F \circ (\sigma \times id)\big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$

where F = homotopy from f to g.

and $\Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$

exercise: $\partial P = g_\# - f_\# - P\partial$ (like proof that $\partial_n \circ \partial_{n+1} = 0$).

▨

The relationship $\partial P + P\partial = g_\# - f_\#$ is expressed as:

P is a chain homotopy from $f_\#$ to $g_\#$

Prop: Chain homotopic maps between exact sequences induce the same map on homology.

Exact Sequences

A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

is <u>exact</u> if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$

<u>chain complex</u> if $\operatorname{im} \alpha_{n+1} \subseteq \ker \alpha_n$ i.e. $\alpha_n \circ \alpha_{n+1} = 0$.

Facts:
(i) $0 \longrightarrow A \xrightarrow{\alpha} B \iff \alpha$ injective
(ii) $A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ surjective
(iii) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ isomorphism.
(iv) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \iff C \cong B/A$.

"short exact sequence"

Collapsing a Subcomplex

<u>Theorem</u>. $(X, A) =$ CW-pair.

There is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}(X/A)$$

$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{q_*} \tilde{H}_{n-1}(X/A) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0.$$

where $i : A \hookrightarrow X$, $q : X \to X/A$.

<u>Cor</u>: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

<u>Proof</u>: Induction on n.

$\tilde{H}_0(S^0) \cong \mathbb{Z}$ ✓

For $n > 0$: $(X, A) = (D^n, S^{n-1}) \rightsquigarrow X/A \cong S^n$.

By theorem:

$$\cdots \longrightarrow \tilde{H}_i(D^n) \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \tilde{H}_{i-1}(D^n) \longrightarrow \cdots$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$ ▨

To prove the Theorem, will do something more general...

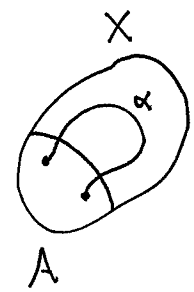
Relative Homology

$A \subseteq X \rightsquigarrow C_n(X,A) \cong C_n(X)/C_n(A)$

Since ∂ takes $C_n(A)$ to $C_{n-1}(A)$, have chain complex

$$\cdots \to C_n(X,A) \to C_{n-1}(X,A) \to \cdots$$

$\rightsquigarrow$ relative homology groups $H_n(X,A)$.



Elements of $H_n(X,A)$ are rep by relative cycles:

$$\alpha \in C_n(X) \quad \text{s.t.} \quad \partial\alpha \in C_{n-1}(A)$$

A relative cycle is trivial in $H_n(X,A)$ iff it is a relative boundary:

$$\alpha \in C_n(X) \quad \alpha = \partial\beta + \gamma \quad \text{some } \beta \in C_{n+1}(X), \gamma \in C_n(A)$$

Will show: $H_n(X,A) \cong H_n(X/A)$.

Goal: Long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A)$$
$$\to H_{n-1}(A)$$

Proof is "diagram chasing".

To start:

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_*"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n(X) \arrow[r, "q_*"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_*"] & C_{n-1}(X) \arrow[r, "q_*"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow$ Short exact sequence of chain complexes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d, "i"] & C_n(A) \arrow[r] \arrow[d] & C_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "q"] & C_n(X) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X,A) \arrow[r] \arrow[d] \arrow[ur, dashed, bend right] & C_n(X,A) \arrow[r] \arrow[d] \arrow[ur, dashed, bend right] & C_{n-1}(X,A) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Commutativity of squares $\Rightarrow$ i_*, q_* chain maps.

$\rightsquigarrow$ induced maps on homology.

Need to define $\partial : H_n(X,A) \longrightarrow H_{n-1}(A)$

Let $c \in C_n(X,A)$ a cycle.

$c = q(\tilde{c})$ $\quad \tilde{c} \in C_n(X)$

$\partial\tilde{c} \in \ker q$ by commutativity.

$\Rightarrow \tilde{c} = i(a)$ some $a \in C_{n-1}(A)$ by exactness.

and $\partial a = 0$ by commut: $i\partial(a) = \partial i(a) = \partial\partial(\tilde{c}) = 0$.

i inj.

Set $\partial[c] = [a] \in H_{n-1}(A)$.

Claim: $\partial : H_n(X,A) \to H_{n-1}(A)$ is a well-defined homomorphism.

Well-defined. · a determined by $\partial\tilde{c}$ since i injective

· different choice $\tilde{c}'$ for $\tilde{c}$ would have

$\tilde{c}' - \tilde{c} \in C_n(A)$, i.e. $\tilde{c}' = \tilde{c} + i(a')$

$\Rightarrow$ a changes to $a + \partial a'$

since $i(a+\partial a') = i(a) + i(\partial a') = \partial\tilde{c} + \partial i(a') = \partial(\tilde{c} + i(a'))$

· different choice for c in $[c]$ is of form $c + \partial c'$

Since $c' = q(b')$ some b' $\leadsto$ $c + \partial c' = c + \partial j(b')$

$b' \leadsto \tilde{c}'$

$= c + q(\partial b') = q(\tilde{c} + \partial b')$

so $\tilde{c}$ replaced by $\tilde{c} + \partial b'$

$\leadsto$ $\partial\tilde{c}$ unchanged.

Homomorphism. Say $\partial[c_1] = [a_1]$, $\partial[c_2] = [a_2]$ via $\tilde{c}_1, \tilde{c}_2$.

Then $q(\tilde{c}_1 + \tilde{c}_2) = c_1 + c_2$

$i(a_1 + a_2) = \partial(\tilde{c}_1 + \tilde{c}_2)$

so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$ //

Theorem. The following sequence is exact:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Proof. More diagram chasing. We'll do 2 of the 6 inclusions needed.

$\operatorname{Im}\partial \subset \ker i_*$ i.e. $i_*\partial = 0$:

$i_*\partial$ takes $[c]$ to $[\partial\tilde{c}] = 0$.

$\ker i_* \subset \operatorname{Im}\partial$: Say $a \in C_{n-1}(A)$, $a \in \ker i_* \Rightarrow i(a) = \partial b$ $b \in C_n(X)$

$\Rightarrow q(b)$ a cycle since $\partial q b = q \partial b = q i(a) = 0$.

& ∂ takes $[q(b)]$ to $[a]$. ▨

Exact Sequences

A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

is <u>exact</u> if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$

<u>chain complex</u> if $\operatorname{im} \alpha_{n+1} \subseteq \ker \alpha_n$ i.e. $\alpha_n \circ \alpha_{n+1} = 0$.

Facts:
(i) $0 \longrightarrow A \xrightarrow{\alpha} B \iff \alpha$ injective
(ii) $A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ surjective
(iii) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ isomorphism.
(iv) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \iff C \cong B/A$ "short exact sequence".

Four Theorems

① Long exact seq. for collapsing subcomplex.
(1') Long exact seq. for pair
③ Excision
② Mayer-Vietoris.

Collapsing a Subcomplex

Theorem. $(X, A) =$ CW-pair.

① There is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}(X/A)$$
$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{q_*} \tilde{H}_{n-1}(X/A) \longrightarrow \cdots$$
$$\cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0.$$

where $i : A \hookrightarrow X$, $q : X \to X/A$.

Cor: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

Proof: Induction on n.
$\tilde{H}_0(S^0) \cong \mathbb{Z}$ ✓
For $n > 0$: $(X, A) = (D^n, S^{n-1}) \rightsquigarrow X/A \cong S^n$.
By theorem:

$$\cdots \longrightarrow \tilde{H}_i(D^n) \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \tilde{H}_{i-1}(D^n) \longrightarrow \cdots$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$ ▨

To prove the Theorem, will do something more general...

Cor (Brouwer Fixed Pt Thm): Every $f : D^n \to D^n$ has a fixed point.

Proof: If not, exists retraction $r : D^n \to \partial D^n$
Consider $\tilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(\partial D^n)$
composition is id & 0 contradiction. ▨

Relative Homology

$A \subseteq X \rightsquigarrow C_n(X,A) \cong C_n(X)/C_n(A)$

Since ∂ takes $C_n(A)$ to $C_{n-1}(A)$, have chain complex

$$\cdots \to C_n(X,A) \to C_{n-1}(X,A) \to \cdots$$

$\rightsquigarrow$ relative homology groups $H_n(X,A)$.

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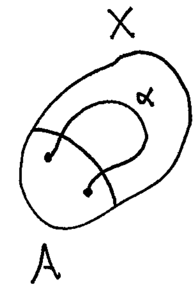

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$i(a_1 + a_2) = \partial(\tilde{c}_1 + \tilde{c}_2)$
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& ∂ takes $[q(b)]$ to $[a]$ ∎

Some facts about relative homology.

<u>Prop</u>: $H_n(X,A) = 0 \ \forall n \iff H_n(A) = H_n(X) \ \forall n$.

Can define reduced relative homology

$\rightsquigarrow \tilde{H}_n(X,A) = H_n(X,A)$ whenever $A \neq \emptyset$.

<u>Prop</u>: If $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs then $f_* = g_*$.

For triples $B \subseteq A \subseteq X$, have

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

and so:

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

Then spectral sequences.

MAYER-VIETORIS

<u>Theorem</u> ② $A, B \subseteq X$ interiors cover X. There is long exact seq:

$$\cdots \to H_n(A\cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

$$x \mapsto x \oplus -x$$

$$x \oplus y \mapsto x - y$$

$$x = x_A + x_B \mapsto \partial x_A$$

- Reduced version formally identical.
- Mayer-Vietoris is abelian version of Van Kampen: For $A\cap B$ path conn

$$MV \Rightarrow H_1(X) = H_1(A) \oplus H_1(B) / H_1(A\cap B)$$

<u>Examples</u> ① $X = S^n$ $A, B =$ (neighborhoods of) hemispheres

$$\tilde{H}_i(A) \oplus \tilde{H}_i(B) = 0 \quad \forall i.$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

② $X =$ Klein bottle $A, B =$ (nbhds of) Möbius bands

$A, B, A\cap B \simeq S^1 \rightsquigarrow$

$$0 \to H_2(X) \to H_1(A\cap B) \to H_1(A) \oplus H_1(B) \to H_1(K) \to 0$$

$$1 \mapsto 2 \oplus -2$$

$\Rightarrow H_2(K) = 0$

$$H_1(K) \cong H_1(A) \oplus H_1(B) / H_1(A\cap B) = (1,0) \oplus (1,1) / (-2,2)$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Excision

Theorem. Let $Z \subseteq A \subseteq X$ closure $Z \subseteq$ interior A
3 Then $(X-Z, A-Z) \hookrightarrow (X,A)$
induces an isomorphism on homology.

Equivalently: $A, B \subseteq X$, interiors cover X.
$(B, A\cap B) \hookrightarrow (X,A)$ induces $\cong$ on H_*
translation $B = X - Z$, $Z = X - B$.

Application: Invariance of ~~Domain~~ Dimension

Theorem: If nonempty open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ are homeomorphic, then $m=n$.

Proof: Let $x \in U$. $H_k(U, U-x) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - x)$ by Excision.
Long exact seq. for $(\mathbb{R}^m, \mathbb{R}^m - x)$:

$$\cdots \to H_k(\mathbb{R}^m) \to H_k(\mathbb{R}^m, \mathbb{R}^m - x) \to H_{k-1}(\mathbb{R}^m - x) \to H_{k-1}(\mathbb{R}^m) \to \cdots$$

$$\Rightarrow \quad H_k(\mathbb{R}^m, \mathbb{R}^m - x) \cong H_{k-1}(\mathbb{R}^m - x)$$

But $H_{k-1}(\mathbb{R}^m - x) \cong H_{k-1}(S^{m-1})$ since $\mathbb{R}^m - x$ def. ret to S^{m-1}
Thus:

$$H_k(U, U-x) = \begin{cases} \mathbb{Z} & k=m \\ 0 & \text{o.w.} \end{cases}$$

In other words, can detect m from homology groups. ∎

Excision also used to show $H_n(X,A) \cong \tilde{H}_n(X/A)$, so Theorem 2 implies Theorem 1. See Hatcher Prop 2.22

Remains to prove Excision and Mayer-Vietoris.

Idea: Subdivide.

Another homology: X = space

$\mathcal{U} = \{U_j\}$ collection of subspaces whose interiors cover X.

$C_n^{\mathcal{U}}(X)$ = chains $\sum n_i \sigma_i$ so each σ_i has image in some U_j

$\partial(C_n^{\mathcal{U}}(X)) \subseteq C_{n-1}^{\mathcal{U}}(X) \rightsquigarrow$ chain complex

$\rightsquigarrow H_n^{\mathcal{U}}(X)$

<u>Prop</u>: $H_n^{\mathcal{U}}(X) \cong H_n(X)$

Specifically, there is a subdivision operator $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ that is a chain homotopy inverse to $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$.

<u>Proof of Excision</u>. To show $H_n(B, A\cap B) \cong H_n(X, A)$.

Let $\mathcal{U} = \{A, B\}$

Note $C_n^{\mathcal{U}}(A)$ naturally identified with $C_n(A)$. by ρ and ι.

$$\Rightarrow \quad C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A) \longrightarrow C_n(X)/C_n(A)$$

induces isomorphism $H_n^{\mathcal{U}}(X, A) \cong H_n(X, A)$.

But:
$$C_n(B)/C_n(A\cap B) \longrightarrow C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$$

obviously an isomorphism: both are free on simplices lying in B but not A. So $H_n(B, A\cap B) \cong H_n^{\mathcal{U}}(X, A)$.

$\blacksquare$

Proof of Mayer-Vietoris. Recall $X = A \cup B$.
Let $\mathcal{U} = \{A, B\}$

There is a short exact seq. of chain complexes:

$$0 \longrightarrow C_n(A \cap B) \longrightarrow C_n(A) \oplus C_n(B) \longrightarrow C_n^{\mathcal{U}}(X) \longrightarrow 0$$

$$x \longmapsto x \oplus -x$$

$$x \oplus y \longmapsto x + y$$

$\rightsquigarrow$ long exact seq. in homology as before.

Substituting $H_n(X)$ for $H_n^{\mathcal{U}}(X)$ (Proposition)

$\rightsquigarrow$ Mayer-Vietoris sequence. $\blacksquare$

A description of $\partial : H_n(X) \longrightarrow H_{n-1}(A \cap B)$:

$\alpha \in H_n(X)$ rep. by cycle z

$z = x + y \quad x \in C_n(A),\ y \in C_n(B)$

$\partial x = -\partial y$ since $\partial z = 0$.

Set $\partial \alpha = \partial x$.

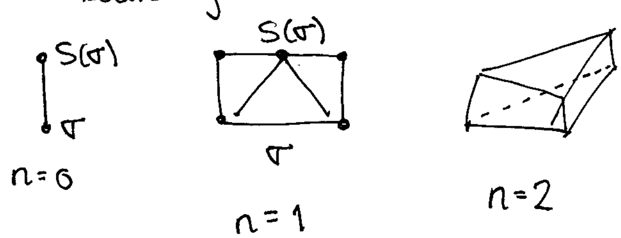
Proof of Prop. Let S = barycentric subdivision.

First show S is a chain homotopy equiv.

then take $\rho = S^N$.

Want $T : C_n(X) \longrightarrow C_{n+1}(X)$ s.t. $T\partial + \partial T = S - id$.

i.e. for any n-simplex σ want $(n+1)$-chain $T\sigma$ with boundary $S(\sigma) - \sigma - T\partial\sigma$



Do $n=1$ case on all 3 sides. Then join all simplices to barycenter on top.

4+ pages in Hatcher!

Homology and Fundamental Group

In many examples, can see $H_1(X) = \pi_1(X)^{ab}$,
e.g. surfaces, $S^1 \vee S^1$, S^n

<u>Theorem</u>. $H_1(X) = \pi_1(X)^{ab}$

<u>Proof</u>. Regarding loops as 1-cycles, there is a map
$$h : \pi_1(X) \longrightarrow H_1(X)$$
To show h a well-defined, surjective homomorphism with kernel $[\pi_1(X), \pi_1(X)]$

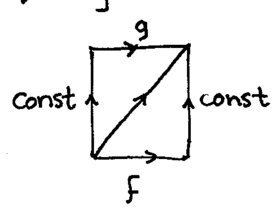
Write $\simeq$ for homotopy, $\sim$ for homology.

Fact 1. $\text{const} \sim 0$

Pf. $H_1(pt) = 0$ also: const loop $= \partial$ const. 2-simplex

Fact 2. $f \simeq g \Rightarrow f \sim g$

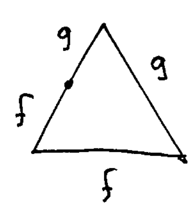
Pf.



boundary $= f - g$

Fact 3. $f \cdot g \sim f + g$

Pf.



boundary $= g - f\cdot g + f$

Fact 4. $\bar{f} \sim -f$

Pf. $f + \bar{f} \overset{③}{\sim} f \cdot \bar{f} \overset{②}{\sim} \text{const} \overset{①}{\sim} 0$

Well-defined. Facts 2 and 3.

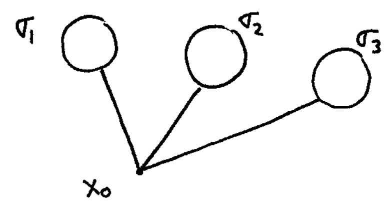
Surjective. Let $\sum n_i \sigma_i$ = 1-cycle

Relabel. $\sum \pm \sigma_i$

By Fact 4, rewrite as $\sum \sigma_i$

Use Fact 3 to organize into loops, relabel $\sum \sigma_i$

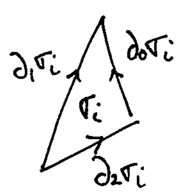
Use Facts 3 and 4 to combine into one loop σ:



The loop σ is in image of h.

Note $[\pi_1(X), \pi_1(X)] \subseteq \ker h$ since $H_1(X)$ abelian.

So say $h(f) \sim 0$. To show $f \in [\pi_1(X), \pi_1(X)]$, i.e. $f = 0$ in $\pi_1(X)^{ab}$.



$$h(f) \sim 0 \Rightarrow f = \partial\left(\sum \sigma_i\right) \qquad \sigma_i = \text{singular } 2\text{-simplex}$$
$$= \sum(\partial_0\sigma_i - \partial_1\sigma_i + \partial_2\sigma_i)$$

Modify all σ_i by homotopy so all vertices map to basepoint for $\pi_1(X) \Rightarrow$ Can regard the sum in $\pi_1(X)^{ab}$

In $\pi_1(X)$ have $(\partial_2\sigma_i)\cdot(\partial_0\sigma_i) = (\partial_1\sigma_i)$ see picture

$\Rightarrow$ each term of sum is 0 in $\pi_1(X)^{ab}$ ∎

Alternate ending. Want to show

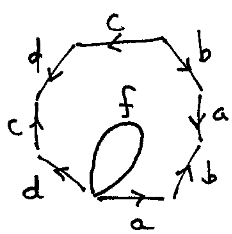
$$h(f) = 0 \Rightarrow f \in [\pi_1(X), \pi_1(X)]$$

$$h(f) = 0 \Rightarrow f = \partial \Sigma \sigma_i$$

Claim: $\Sigma \sigma_i$ represents an orientable surface with one boundary, namely f.

Pf: Adjacent triangles must have both ∂'s clockwise or both counterclockwise.

Classification of surfaces $\Rightarrow$ $\Sigma \sigma_i$ is



$\Rightarrow$ f a product of g commutators. ▨

Some History

An n-manifold is a Hausdorff space where each point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Poincaré's First Conjecture. If X is a 3-manifold with $H_1(X) = 0$, then X is homeomorphic to S^3.

Counterexample: Poincaré Dodecahedral Space.

Take a solid dodecahedron, glue opposite faces with $2\pi/10$ clockwise twist. This has same homology as S^3 ("homology sphere")

This led Poincaré to develop π_1. $\rightsquigarrow |\pi_1(PDS)| = 120$.

The last theorem shows π_1 has more information than H_1. Sometimes this is important information!

Applications of Homology

① Jordan Curve Theorem, etc.

homeo onto image. in this case, any injective continuous map.

<u>Theorem</u>. Let $h: S^1 \to \mathbb{R}^2$ embedding.
Then $\mathbb{R}^2 - h(S^1)$ has exactly 2 connected components.

Easy for nice curves (e.g. polygonal). Must consider things like Osgood curves, which have positive (exterior) area (these are obtained by perturbing space filling curves).

<u>Prop</u>: (a) If $h: D^k \to S^n$ an embedding, then
$$\tilde{H}_i(S^n - h(D^k)) = 0 \quad \forall i$$
(b) If $h: S^k \to S^n$ an embedding, $k < n$, then
$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise} \end{cases}$$

(b) implies any S^{n-1} in S^n divides S^n into two components, each with homology of a point.
For $n=2$, Jordan Curve Thm.
For $n=3$, it is possible for one component to be not simply connected. (Alexander horned sphere.)

(b) also implies $H_1(S^3 - \text{knot}) \cong \mathbb{Z}$.

Proof of Prop: (a) Induct on k

$k=0 \rightsquigarrow S^n - h(D^k) \cong \mathbb{R}^n$ ✓

Replace D^k with I^k.

Let $A = S^n - h(I^{k-1} \times [0, 1/2])$

$B = S^n - h(I^{k-1} \times [1/2, 1])$

Induction $\Rightarrow \tilde{H}_i(A \cup B) = \tilde{H}_i(S^n - h(I^{k-1} \times 1/2)) = 0.$

Mayer-Vietoris $\Rightarrow$

$$\Phi : \underset{\substack{\| \\ S^n - h(D^k)}}{\tilde{H}_i(A \cap B)} \longrightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B) \quad \text{isomorphism } \forall i.$$

So if $[\alpha] \neq 0$ in $\tilde{H}_i(S^n - h(D^k))$ then

$\alpha \neq 0$ in $H_i(S^n -$ half of $h(I^k))$

Say these halves converge to $I^{k-1} \times \{p\}$.

By above, α a boundary in $\tilde{H}_i(S^n - h(I^{k-1} \times \{p\}))$

Say $\alpha = \partial \beta$.

β compact $\Rightarrow [\alpha] = 0$ at some finite stage.

$\rightsquigarrow$ contradiction. ▨

(b) Induct on k.

$k=0 \rightsquigarrow S^n - h(S^0) \simeq S^{n-1} \times \mathbb{R}$ ✓

Let $S^k = D^k_+ \cup_{S^{k-1}} D^k_-$

$A = S^n - h(D^k_+)$, $B = S^n - h(D^k_-)$

Mayer-Vietoris plus (a) $\Rightarrow$

$$\tilde{H}_{i+1}(S^n - h(S^{k-1})) \cong \tilde{H}_i(S^n - h(S^k))$$ ▨

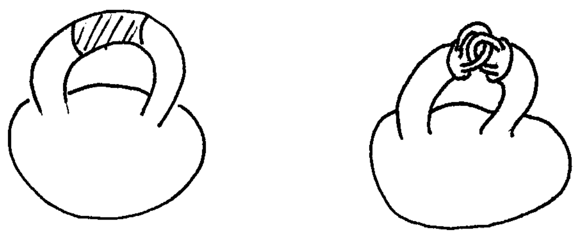
Exercise. Examine the case $k=n$

$\rightsquigarrow S^n$ cannot embed in $\mathbb{R}^n$

$\mathbb{R}^m$ cannot embed in $\mathbb{R}^n$ $m > n$.

Aside: Alexander Horned Sphere

The Alexander Horned Ball is the intersection $\bigcap_{i=1}^{\infty} X_i$



X_1 X_2

$$\pi_1(AHB^c) = \langle \alpha_0, \alpha_1, \ldots \mid [\alpha_1, \alpha_2] = \alpha_0, \; [\alpha_3, \alpha_4] = \alpha_1, \; [\alpha_5, \alpha_6] = \alpha_2, \; \ldots \rangle$$

This group is nontrivial – it is an increasing union of free groups. But since each α_i is a commutator, the abelianization is trivial.

② Invariance of Domain

Theorem U open in $\mathbb{R}^n$, $h: U \to \mathbb{R}^n$ embedding
$\Rightarrow h(U)$ open in $\mathbb{R}^n$.

Proof Think of $\mathbb{R}^n$ as S^n - pt.
Equivalent to show $h(U)$ open in S^n.
Let $x \in U$, D^n = disk about x in U.
Suffices to show $h(\text{int } D^n)$ open in S^n
Prop (b) $\Rightarrow S^n - h(D^n)$ has 2 path components.
The components are $h(\text{int } D^n)$, $S^n - h(D^n)$. Indeed:
· Since $h(\text{int } D^n)$ path conn, these sets are disjoint
· $S^n - h(D^n)$ path conn by Prop (b)
Since $S^n - h(\partial D^n)$ open in S^n ($h(\partial D^n)$ compact in Hausdorff),
its path components = connected components (true for loc. comp.)
An open set with finitely many comp. must have
each comp. open
$\Rightarrow h(\text{int } D^n)$ open in $S^n - h(\partial D^n)$
$\Rightarrow$ open in S^n ▨

Cor: M = compact n-manifold, N = connected n-manifold
Then any embedding $M \xrightarrow{h} N$ is surjective, hence a homeo.

Proof: $h(M)$ closed in N (compact in Hausdorff)
Since N conn, suffices to show $h(M)$ open in N.
Let $x \in M$. Choose neighborhood V of $h(x)$ homeo to $\mathbb{R}^n$.
Choose nbhd U of x in $h^{-1}(V)$ homeo to $\mathbb{R}^n$
$h|_U$ an embedding into V. Thm $\Rightarrow h(U)$ open in V,
hence open in N. ▨

③ Division Algebras

An algebra over $\mathbb{R}$ is $\mathbb{R}^n$ with bilinear multiplication

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(a, b) \mapsto ab$$

So: $a(b+c) = ab+ac$, $(a+b)c = ac+bc$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$

It is a division algebra if $ax=b$, $xa=b$ always solvable for $a \neq 0$. ("no zero divisors")

Four classical examples: $\mathbb{R}$, $\mathbb{C}$, Quaternions, Octonians

Theorem. $\mathbb{R}$ & $\mathbb{C}$ are the only finite dimensional division algebras over $\mathbb{R}$ that are commutative and have id.

Proof. We'll show: a fin. dim. comm. div alg. has dim ≤ 2.
Suppose $\mathbb{R}^n$ has a comm. div. alg. structure.
Define $f: S^{n-1} \to S^{n-1}$ by $f(x) = x^2/|x^2|$
$\rightsquigarrow$ induced map $\bar{f}: \mathbb{R}P^{n-1} \to S^{n-1}$
Claim: $\bar{f}$ injective

Pf: $f(x) = f(y) \Rightarrow x^2 = \alpha^2 y^2 \Rightarrow x^2 - \alpha^2 y^2 = 0$
$\Rightarrow (x+\alpha y)(x-\alpha y) = 0$ (commutativity)
No zero div $\Rightarrow x = \pm\alpha y \Rightarrow x = \pm y$.

$\bar{f}$ injective map of compact Hausdorff $\Rightarrow$ embedding
Cor $\Rightarrow \bar{f}$ surjective if $n > 1$.
$\Rightarrow \mathbb{R}P^{n-1} \cong S^{n-1} \Rightarrow n = 2$ (compare π_1)

A little more algebra to get full theorem. ▨

DEGREE

$$f : S^n \to S^n \rightsquigarrow f_* : H_n(S^n) \to H_n(S^n)$$
$$\alpha \mapsto d\alpha$$

d = degree of f.

Facts (i) $\deg \mathrm{id} = 1$
(ii) $\deg f = 0$ if f not surjective
(iii) $\deg f = \deg g \Leftrightarrow f \simeq g$ $\Rightarrow$ due to Hopf.
(iv) $\deg fg = \deg f \deg g$
(v) $\deg f = -1$ f = reflection along equator
(vi) $\deg(\text{antipodal}) = (-1)^n$

④ Hairy Ball Theorem

Theorem. S^n has a continuous field of nonzero tangent vectors iff n is odd.

Proof. ($\Rightarrow$) Let $v(x)$ = vector field on S^n. Translate $v(x)$ to origin
$\rightsquigarrow v(x) \perp x$ in $\mathbb{R}^{n+1}$
$v(x) \neq 0 \ \forall x \rightsquigarrow$ can replace $v(x)$ with $v(x)/|v(x)|$
$\Rightarrow (\cos t)x + (\sin t)v(x)$ = unit S^1 in $x, v(x)$ plane
$f_t(x) = (\cos t)x + (\sin t)v(x)$ a homotopy from id ($t=0$) to antipodal map ($t=\pi$)
(iii) $\Rightarrow \deg \mathrm{id} = \deg \mathrm{antip}$.
(i), (vi) $\Rightarrow n$ odd.

($\Leftarrow$) For $n = 2k-1$ set $v(x_1, \dots, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$. ▨

One more fact about degree:

(vi) If f has no fixed points, then $\deg f = (-1)^{n+1}$

proof: find homotopy to antipodal map (straight line)

⑤ Prop: $\mathbb{Z}/2\mathbb{Z}$ is only group that can act freely on S^n if n is even.

Pf: Say $G \curvearrowright S^n \rightsquigarrow d: G \to \{\pm 1\}$ homomorphism by (iv)

Action free $\Rightarrow d(g) = (-1)^{n+1}$ $g \neq id$ by (vi)

n even $\Rightarrow \ker d = 1 \Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$. ▨

Can also use degree to define/compute cellular homology

$\rightsquigarrow$ compute homology of $\mathbb{C}P^n$, $S^n \times S^n$, T^n, $\mathbb{R}P^n$, $L(p,q)$, etc.

see text.

⑥ Borsuk-Ulam Theorem

Prop: Say $f: S^n \to S^n$, $f(-x) = -f(x)$ $\forall x$ (odd map).
Then f has odd degree.

Theorem: $g: S^n \to \mathbb{R}^n \Rightarrow \exists x$ s.t. $g(x) = g(-x)$.

Proof: Let $f(x) = g(x) - g(-x)$, say $f(x) \neq 0$ $\forall x$.
Replace $f(x)$ by $f(x)/|f(x)|$
$\rightsquigarrow f: S^n \to S^{n-1}$ odd
Prop $\Rightarrow f|_{\text{equator}}$ has odd degree.
But either hemisphere gives a nullhomotopy.
CONTRADICTION. ▨

(17) Lefschetz Fixed Point Theorem

Trace: for $\varphi : A \to A$ $\quad A =$ f.g. abelian group

$$\operatorname{tr}\varphi = \operatorname{tr}(A/\text{torsion} \longrightarrow A/\text{torsion})$$

$X =$ space with finitely generated homology, trivial H_i $\ i \geq N$.
e.g. finite simplicial complex.

The <u>Lefschetz number</u> of $f : X \to X$ is

$$\tau(f) = \Sigma(-1)^i \operatorname{tr}(f_* : H_i(X) \longrightarrow H_i(X))$$

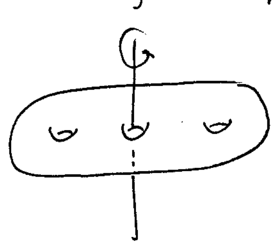
<u>Theorem</u> $\tau(f) =$ sum of indices of fixed points $\quad$ (assume fixed points are isolated)

In particular $\tau(f) \neq 0 \Rightarrow$ fixed points
Brouwer FPT is corollary.

~~Recall~~ Index of fixed point p is $\deg(\bar{f} : (X, X-p) \longrightarrow (X, X-p))$

<u>Linear maps</u>. Modulo torsion, $\mathbb{R}P^n$ n even has homology of pt.
$\Rightarrow$ every map has a fixed point
$\Rightarrow$ every linear map $\mathbb{R}^n \to \mathbb{R}^n$, n odd
has an eigenvector (can also use elementary reasoning).

Can do many examples of LFPT with surfaces, e.g.



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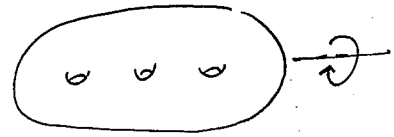


Preparation: Approximation by simplicial maps

Simplicial maps. K, L simplicial complexes
$K \to L$ simplicial if simplices $\mapsto$ simplices, linearly.

<u>Theorem</u>. K = finite simplicial complex, L = simplicial complex.
Any $f: K \to L$ is homotopic to a map that is simplicial w.r.t. some subdivision of K.

<u>Idea</u> <u>of</u> <u>Proof</u> that $\tau(f) \neq 0 \Rightarrow \exists$ fixed points.

Assume $f: X \to X$ has no fixed points
Simplicial approx $\rightsquigarrow$ $g: X \to X$ simplicial, homotopic to f
$g(\sigma) \cap \sigma = \emptyset$ $\forall$ simplices σ.

Note $\tau(f) = \tau(g)$.
To show $tr(g_*) = 0$ in all dim.

Key: $\tau(g) = \sum (-1)^n \, tr(g_*: H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1}))$
Use the fact that g takes X^n to X^n plus some algebra.

Since g crushes/permutes cells without fixing any, all of these traces are 0. $\blacksquare$

Cohomology

Same basic information as homology, but get

- multiplicative structure
- pairing with homology
- contravariance

Quick idea: $X = \Delta$-complex

G = abelian group, say $\mathbb{Z}$

$\Delta^i(X)$ = functions from i-simplices of X to G.

= homomorphisms $\Delta_i(X) \to G$

$\delta : \Delta^i(X,G) \to \Delta^{i+1}(X,G)$ coboundary

For $f \in \Delta^i$, σ an $(i+1)$-simplex, $\delta f(\sigma) = \sum (-1)^k f(\partial_k \sigma)$

$H^*(X;G)$ is homology of this chain complex

~~Examples~~ Graphs. X = 1-dim Δ-complex = oriented graph

Let $f \in \Delta^0(X,G)$

$\delta f(e) = f(v_t) - f(v_0)$

= change of f over e "derivative"

think: f = elevation

$\rightsquigarrow$ chain complex:

$$0 \to \Delta^0(X,G) \xrightarrow{\delta} \Delta^1(X,G) \to 0$$

$H^0(X,G) = \ker \delta$

= functions constant on each component

= direct product of components

(as opposed to direct sum in homology case)

$H^1(X,G) = \Delta^1(X,G)/\mathrm{Im}\,\delta$

So for $f \in \Delta^1(X,G)$, have $[f]=0$ in $H^1(X,G)$ iff f has an antiderivative.

Examples. ① X = tree

Antiderivatives always exist

$\Rightarrow H^1(X,G) = 0$.

② X = 

$\Delta^1(X,G) \cong G$

No nontrivial function has an antiderivative

$\rightsquigarrow H^1(X,G) \cong G$

③ $X = \bigvee_\alpha S^1$

$\rightsquigarrow H^1(X,G) \cong \prod_\alpha G$

More generally. X = any ~~tree~~ graph.

Let T = maximal tree (or forest), E = edges outside T

$\rightsquigarrow H^1(X,G) = \prod_E G$ (again, instead of direct sum).

Why? First consider $\{f \mid f|_T = 0\}$

Two of these are cohomologous $\Leftrightarrow$ they are equal (only possible antiderivative is F = const).

Next show any $f' \in \Delta^1$ is cohomologous to some f with $f|_T = 0$. Modify f' by making one edge e of T evaluate to 0, say add g to $f'(e)$. Then for any edge e' of $X-T$, either add or subtract g, depending on whether loop through e,e' traverses them in same or diff directions. Check new f' cohomologous to old.

Two dimensions. $X = 2\text{-dim } \Delta\text{-complex}$

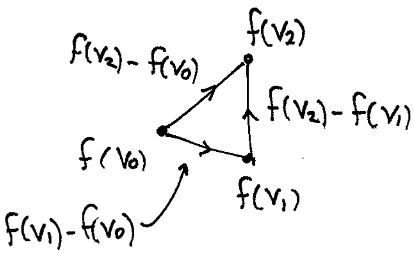
$$\delta : \Delta^1(X,G) \to \Delta^2(X,G)$$

$$\delta f([v_0,v_1,v_2]) = f([v_1,v_2]) - f([v_0,v_2]) + f([v_0,v_1])$$

Check that

$$0 \to \Delta^0(X,G) \to \Delta^1(X,G) \to \Delta^2(X,G) \to 0$$

is a chain complex: say $f \in \Delta^0(X,G)$.



$$\delta\delta f([v_0,v_1,v_2]) = (f(v_1) - f(v_0)) + (f(v_2) - f(v_1)) - (f(v_2) - f(v_0))$$

i.e. if you hike a loop, total elevation change is zero.

1-cocycles: $\delta f = 0$ iff

$$f([v_0,v_2]) = f([v_0,v_1]) + f([v_1,v_2])$$

so δf measures failure of additivity.

This is the <u>local</u> obstruction to f being in $\operatorname{im}\delta$

And $f \neq 0$ in $H^1(X) \iff$ does not come from $F \in \Delta^0$.

i.e. if there is a <u>global</u> obstruction.

Analogue with calculus. 1-forms on $\mathbb{R}^3 \longleftrightarrow$ vector fields

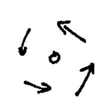
Want to know if vector field is ∇f

local obstruction: curl $= 0$. (closed)

global obstruction: line integrals $= 0$. (exact)

In $\mathbb{R}^n$, all closed forms are exact.

Not true in other spaces, e.g. $\mathbb{R}^2 - \{0\}$



de Rham cohomology: closed forms / exact forms.

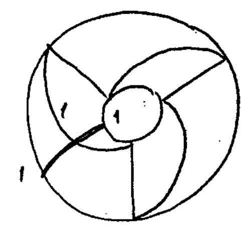
Geometric interpretation of 1-cocycles, X a surface.

Take $G = \mathbb{Z}_2$. $\delta f = 0$ means f takes value 1 on even # of edges in each Δ.

$\rightsquigarrow$ collection of curves, arcs

$[f] = 0 \iff$ can color regions black & white.

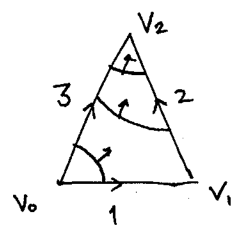
examples. disk, annulus:



unlabeled $= 0$.

Take $G = \mathbb{Z}$. Again $\delta f = 0 \longrightarrow$ collection of curves

e.g.



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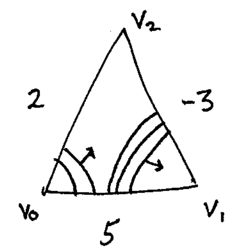


arrows point up.

$[f] = 0 \iff$ can assign elevation to each vertex consistently.

exercise. Construct nontrivial cocycle on annulus.

So: in annulus, can walk in a loop and change your elevation!

cf. international dateline.

Exercise: Find geometric interpretations of 1- & 2-cocycles in a 3-manifold.

Cohomology Groups (Some Abstract Algebra)

Start with a chain complex of abelian groups C:

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

$$\rightsquigarrow H_n(C) = \ker\partial_n / \operatorname{im}\partial_{n+1}$$

To get cohomology, we dualize: replace each C_n with its dual

$$C_n^* = \operatorname{Hom}(C_n, G)$$

replace each ∂ with $\delta = \partial^* : C_{n-1}^* \to C_n^*$

Notice: $\delta\delta = \partial^*\partial^* = (\partial\partial)^* = 0^* = 0.$

$$\rightsquigarrow H^n(C,G) = \ker\delta / \operatorname{im}\delta$$

Guess: $H^n(C,G) \cong \operatorname{Hom}(H^n(C), G)$ Too optimistic, but almost true. It is true for graphs.

Example.

$$\begin{array}{cccccccccc} & & C_3 & & C_2 & & C_1 & & C_0 & \\ C: & 0 \to & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to 0 \end{array}$$

$\rightsquigarrow H_0(C) = \mathbb{Z}$, $H_1(C) = \mathbb{Z}/2\mathbb{Z}$ $H_2(C) = 0$ $H_3(C) = \mathbb{Z}$

use formula $\delta\varphi = \varphi\partial$

$$C^*: \quad 0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

$\rightsquigarrow H^0(C,\mathbb{Z}) = \mathbb{Z}$, $H^1(C,\mathbb{Z}) = 0$, $H^2(C,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $H^3(C,\mathbb{Z}) = \mathbb{Z}$

See: Torsion shifts up one dimension.

This holds in general, since any chain complex of finitely generated free abelian groups splits as a direct sum of $0 \to \mathbb{Z} \to 0$

and $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$

UNIVERSAL COEFFICIENT THEOREM FOR COHOMOLOGY

$C: \quad \cdots \to C_n \to C_{n-1}$ chain complex.

$\rightsquigarrow H_n(C)$

$T_n(C)$ = torsion subgroup of $H_n(C)$.

We just showed: If the $H_n(C)$ are finitely generated, and each C_i is free abelian, then

$$H^n(C,\mathbb{Z}) \cong H_n(C)/T_n(C) \oplus T_{n-1}(C)$$

This is a special case of:

<u>Theorem</u>. There is a split short exact sequence:

$$0 \to Ext(H_{n-1}(C),G) \to H^n(C,G) \to Hom(H_n(C),G) \to 0$$

The group $Ext(H_{n-1}(C),G)$ is explicit. It describes all extensions of $H_{n-1}(C)$ by G. Some properties: If H is finitely gen, then

① $Ext(H \oplus H', G) \cong Ext(H,G) \oplus Ext(H',G)$

② $Ext(H,G) = 0$ if H is free

③ $Ext(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

These imply the special case of UCT above.

Universal coefficient theorem for homology:

$$H_n(X,\mathbb{Q}) \cong H_n(X,\mathbb{Z}) \otimes \mathbb{Q} \qquad \text{(later)}.$$

Cohomology of Spaces

X = space, G = abelian group

$C^n(X,G)$ (= singular n-chains with coefficients in G, except allow ∞ sums)

$\quad = $ dual of $C_n(X)$

$\quad = \mathrm{Hom}(C_n(X), G)$

Coboundary δ is ∂_*: for $\varphi \in C^n(X,G)$

$$\delta\varphi : C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G.$$

Again, $\delta^2 = 0$.

$\rightsquigarrow H^n(X,G)$ cohomology group with coefficients in G.

$$= \ker\delta / \mathrm{im}\,\delta = \text{cocycles}/\text{coboundaries}$$

Cocycles. A cochain φ is a cocycle iff $\delta\varphi = \varphi\partial = 0$,

i.e. φ vanishes on all boundaries.

It is a coboundary if it has an "antiderivative."

Since $C_n(X)$ free, UCT gives:

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X,G) \to \mathrm{Hom}(H_n(X), G) \to 0$$

"Cohomology groups of X with arbitrary coefficients is determined by the homology groups of X with $\mathbb{Z}$ coefficients."

What is Ext?

Let $B_n = \mathrm{im}\,\partial_{n+1}$ (boundaries)

$Z_n = \ker \partial_n$ (cycles)

$\rightsquigarrow i_n : B_n \to Z_n$

$\mathrm{Ext}(H_{n-1}(X), G) = \mathrm{Coker}\; i_{n-1}^*$

dual to i_{n-1}

Cohomology in Low Dimensions

$\underline{\underline{n=0}}$ Ext term is trivial, so

$$H^0(X,G) \cong \mathrm{Hom}(H_0(X), G)$$

Can see directly from definitions:

sing. 0-simplices $\longleftrightarrow$ points of X

cochains $\longleftrightarrow$ functions $X \to G$ (not continuous)

Cocycles $\longleftrightarrow$ vanish on boundaries

$\longleftrightarrow$ const. on each path component

$$\Rightarrow H^0(X,G) = \text{functions } \{\text{path components of } X\} \to G$$
$$= \mathrm{Hom}(H_0(X), G).$$

$\underline{\underline{n=1}}$ $\mathrm{Ext} = 0$ since $H_0(X)$ free

$$\Rightarrow H^1(X,G) \cong \mathrm{Hom}(H_1(X), G)$$
$$\cong \mathrm{Hom}(\pi_1(X), G) \text{ if } X \text{ path conn.}$$

Coefficients in a Field

$H_n(X,F)$ = homology gps of chain complex of F-vector spaces $C_n(X,F)$

Dual complex $\mathrm{Hom}_F(C_n(X,F), F) = \mathrm{Hom}(C_n(X), F)$

$\rightsquigarrow H^n(X,F)$

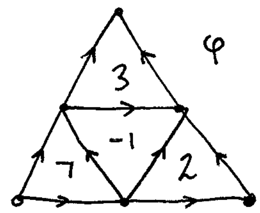
Can generalize UCT to fields (or pid's) $\rightsquigarrow$ Ext vanishes for fields

$$\rightsquigarrow H^n(X,F) \cong \mathrm{Hom}_F(H_n(X,F), F)$$

For $F = \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$, $\mathrm{Hom}_F = \mathrm{Hom}$

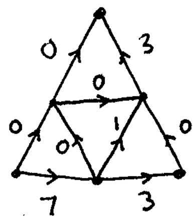
Examples of 2-cocycles

① $X = D^2$



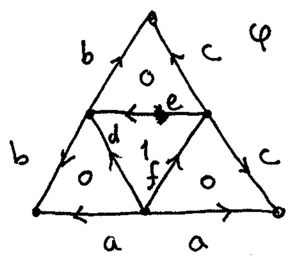
We know $H^2(D^2, \mathbb{Z}) = 0$
so $\varphi = \delta\psi$.
What is ψ?

Solution:



No obstructions.

② $X = S^2$



Want to show $[\varphi] \neq 0$
in $H^2(S^2, \mathbb{Z})$
i.e. no antiderivative ψ.

Any ψ with $\partial\psi = \varphi$ must satisfy:

writing
a for $\psi(a)$

$$b + d = a$$
$$e + c = a$$
$$\left.\begin{array}{l} b + f = c \\ e + f = d + 1 \end{array}\right\} \Rightarrow (b+d) - (e+c) = 1$$
$$\Rightarrow a - a = 1.$$

③ $X = T^3$, $G = \mathbb{Z}/2\mathbb{Z}$.

Realize T^3 as Δ-complex by subdividing cube into 6 tetrahedra, identifying opp faces of the cube. Let L = line segment in cube that is a loop in T^3, misses 1-skeleton. Declare $\varphi(T) = 1$ if $T \cap L \neq \phi$. Show $[\varphi] \neq 0$ in $H^2(T^3, \mathbb{Z}/2\mathbb{Z})$.

Cohomology Theory

Reduced groups, relative groups, long exact seq of pair, excision, Mayer-Vietoris, all work for cohomology.

Induced Homomorphisms - Contravariance

Given $f: X \to Y$, get chain maps $f_\#: C_n(X) \to C_n(Y)$

Dualize: $f^\#: C^n(Y,G) \to C^n(X,G)$

$f_\# \partial = \partial f_\#$ dualizes to $\delta f^\# = f^\# \delta$

$\rightsquigarrow f^*: H^n(Y,G) \to H^n(X,G)$

with: $(fg)^* = g^* f^*$ & $(id)^* = id$

Say $X \mapsto H^n(X,G)$ is a contravariant functor.

Homotopy Invariance

$f \simeq g : X \to Y \Rightarrow f^* = g^* : H^n(Y) \to H^n(X)$.

Dualize the proof for homology

Recall there is a chain homotopy P s.t. $g_\# - f_\# = \partial P + P\partial$

Dualize: $g^\# - f^\# = P^* \delta + \delta P^*$

$\rightsquigarrow P^*$ a chain homotopy between $f^\#$ & $g^\#$

So all the work has been done.

Product Structures

There are three natural products with homology & cohomology:

① Evaluation pairing:

$$H^k(X) \times H_k(X) \longrightarrow \mathbb{Z}$$

Can use this to show cocycles, or cycles, are nontrivial!

② Cup product:

$$H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$

$$(\varphi, \psi) \longmapsto \varphi \cup \psi$$

$\rightsquigarrow H^*(X)$ is a graded ring.

③ Cap product:

$$H^p(X) \times H_n(X) \longrightarrow H_{n-p}(X)$$

$$(\varphi, \alpha) \longmapsto \varphi \cap \alpha$$

Big Goal:

Poincaré Duality Theorem. Let M = compact, connected, oriented n-manifold. Then

$$H^p(M) \longrightarrow H_{n-p}(M)$$

$$\varphi \longmapsto \varphi \cap [M]$$

is an isomorphism.

We have already since examples of p-cocycles in n-manifolds of the form "intersect with this $(n-p)$-cycle". These are Poincaré duals.

Will see: under PD, cap product is intersection.

Cup Product

Want to define a product on $H_*(X)$.

There is a cross product $\quad H_i(X) \times H_j(Y) \longrightarrow H_{i+j}(X \times Y)$

$$(e_i, e_j) \longmapsto e_i \times e_j$$

Taking $X = Y$: $\quad H_i(X) \times H_j(X) \longrightarrow H_{i+j}(X \times X) \xrightarrow{?} H_{i+j}(X)$

Need a natural map $X \times X \to X$.

If X is a group, can multiply $\rightsquigarrow$ Pontryagin product.

Otherwise only natural map is projection $\rightsquigarrow$ stupid product.

For H^*, situation is better. Want

$$\cancel{H^i(X) \times H^j(X) \to H^{i+j}(X) \xrightarrow{?}}$$

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{?} H^{i+j}(X)$$

This requires a natural map $X \to X \times X$ $\rightsquigarrow$ diagonal!

This is the cup product.

We can also define cup product from scratch:

For $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$ $\qquad R = $ ring.

the cup product $\varphi \cup \psi \in C^{k+\ell}(X,R)$ is

given by : $\quad (\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$

for a simplex $\sigma : \Delta^{k+\ell} \to X$.

To show cup product induces a product on cohomology.

<u>Lemma</u> $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

<u>Pf</u> Say $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$, $\sigma : \Delta^{k+\ell+1} \to X$.

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|[v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}])\,\psi(\sigma|[v_{k+1},\ldots,v_{k+\ell+1}])$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|[v_0,\ldots,v_k])\,\psi(\sigma|[v_k,\ldots,\hat{v}_i,\ldots,v_{k+\ell+1}])$$

Last term of first sum cancels first sum of second.
Rest is $\delta(\varphi\cup\psi)(\sigma) = (\varphi\cup\psi)(\partial\sigma)$. ▨

Since $\delta(\varphi\cup\psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi$
$\rightsquigarrow$ product of cocycles is a cocycle.

Also, the product of a cocycle and a coboundary is a coboundary:

$$\psi = \delta\theta, \quad \delta\varphi = 0 \implies \delta(\varphi\cup\theta) = \delta\varphi \cup \theta \pm \varphi \cup \delta\theta$$
$$= \pm \varphi \cup \psi.$$

We thus have an induced cup product

$$H^k(X,R) \times H^\ell(X,R) \xrightarrow{\cup} H^{k+\ell}(X,R)$$

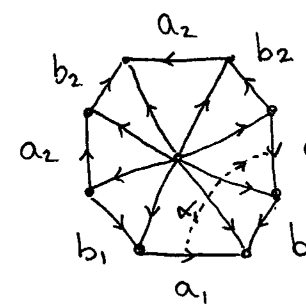
It is associative and distributive, since it is on cochain level.
If R has 1 then $H^*(X,R)$ has identity, namely:
$1 \in H^0(X,R)$ taking value $1 \in R$ on each 0-simplex.

Note: The canonical isomorphism between simplicial/singular H^* preserves $\cup$, so can switch back & forth.

Example: Surfaces

$X = M_g$. Will show $\cup : H^1(M_g,\mathbb{Z}) \times H^1(M_g,\mathbb{Z}) \longrightarrow H^2(M_g,\mathbb{Z}) = \mathbb{Z}$

is algebraic intersection.



a_i, b_i form a basis for $H_1(M_g,\mathbb{Z})$.

UCT $\Rightarrow$ $H^1(M_g) \cong \mathrm{Hom}(H_1(M_g),\mathbb{Z})$

Basis for $H_1 \rightsquigarrow$ dual basis for H^1

$$a_i \rightsquigarrow \varphi_i \quad \begin{array}{l} a_1 \longmapsto 1 \\ \text{others} \longmapsto 0. \end{array}$$

Can represent φ_i, ψ_i by simplicial cocycle $\rightsquigarrow$ dotted arc. α_i, β_i.

α_i evaluates to 1 on an edge like

-1 on an edge like

Compute $\varphi_1 \cup \psi_1$ from definition.

Takes value 0 on all cells but SE, where it takes value 1.

We know $H_2(M_g) = \mathbb{Z} = \langle [M_g] \rangle$ ← fundamental class

UCT $\Rightarrow$ $H^2(M_g,\mathbb{Z}) \cong \mathrm{Hom}(H_2(M_g),\mathbb{Z})$.

So which elt of $H^2(M_g,\mathbb{Z})$ is $\varphi_1 \cup \psi_1$?

We check $(\varphi_1 \cup \psi_1)([M_g]) = 1$

This tells us both that (i) $[M_g]$ generates $H_2(M_g)$

(ii) $\varphi_1 \cup \psi_1$ is dual to $[M_g]$, hence a gen. for $H^2(M_g,\mathbb{Z})$.

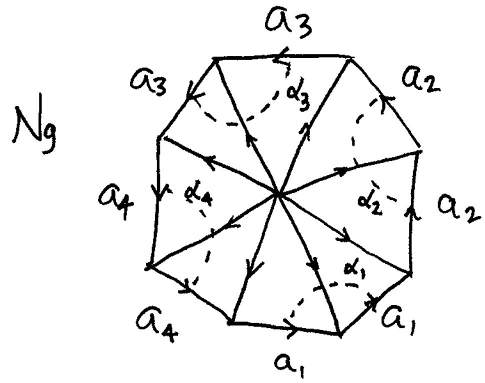
In general, identifying $H^2(M_g,\mathbb{Z})$ with $\mathbb{Z}$:

$$\cup = \hat{\iota}$$

← algebraic intersection.

Suffices to check on generators.

Example: Nonorientable Surfaces



Use $\mathbb{Z}/2\mathbb{Z}$ coefficients since

$H_2(N_g) = 0$

$H_2(N_g; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

Claim: $H^2(N_g, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Pf: Any single Δ gives a cocycle. φ
Any two adjacent triangles are cohomologous
$\rightsquigarrow$ any cocycle is $k\varphi$.

Can also use UCT and $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

Can check: $\alpha_i \cup \alpha_i = 1$
$\alpha_i \cup \alpha_j = 0$

This is again intersection number: if you push off α_i it intersects itself in one point.

The $g=1$ case is $\mathbb{RP}^2$.

$$\rightsquigarrow H^*(\mathbb{RP}^2) = \{1, \alpha, \alpha \cup \alpha\}$$
$$= \mathbb{Z}/2\mathbb{Z}[\alpha]/\langle \alpha^2 \rangle.$$

Naturality

<u>Prop</u>: For $f: X \to Y$, the induced $f^*: H^n(Y,R) \to H^n(X,R)$ satisfies:

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

<u>Pf</u>: Already true on cochain level: $f^\#(\varphi) \cup f^\#(\psi) = f^\#(\varphi \cup \psi)$.

$$\begin{aligned}(f^\#(\varphi) \cup f^\#(\psi))(\sigma) &= f^\#\varphi(\sigma|_{[v_0,\dots,v_k]})\, f^\#\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\ &= \varphi(f\sigma|_{[v_0,\dots,v_k]})\, \psi(f\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\ &= (\varphi \cup \psi)(f\sigma) \\ &= f^\#(\varphi \cup \psi)(\sigma).\end{aligned}$$

▨

Relative Version

$C^k(X,A;R)$ = cochains that vanish on A
(more natural than $C_k(X,A)$ since it is a subgroup, not a quotient).

Have cup products:

$$\left.\begin{array}{l} H^k(X;R) \times H^\ell(X,A;R) \\ H^k(X,A;R) \times H^\ell(X;R) \\ H^k(X,A;R) \times H^\ell(X,A;R) \end{array}\right\} \longrightarrow H^{k+\ell}(X,A;R)$$

And: $H^k(X,A;R) \times H^\ell(X,B;R) \longrightarrow H^{k+\ell}(X, A \cup B; R)$.

The Cohomology Ring

Define $H^*(X,R) = \bigoplus H^k(X,R)$

Elements are finite sums $\Sigma \alpha_i$ with $\alpha_i \in H^i(X,R)$.

The product is $\Sigma \alpha_i \Sigma \beta_j = \Sigma \alpha_i \beta_j$

(writing xy for $x \cup y$).

$\rightsquigarrow$ $H^*(X,R)$ is a ring. It has 1 if R has 1.

We saw: $H^*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) = \{ a_0 + a_1\alpha + a_2\alpha^2 : a_i \in \mathbb{Z}/2\mathbb{Z} \}$

$= \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^3)$ nice!

One can also show: $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$. $|\alpha| = 1$

$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]$

and $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ $|\alpha| = 2$.

$H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[\alpha]$.

H^* is a graded ring, a ring of form $\bigoplus A_k$ with

A_k = additive subgroup, $A_k \times A_\ell \subseteq A_{k+\ell}$.

Write $|\alpha|$ for the degree (i.e. which A_k it lives in).

There are spaces with same H_k & H^k groups, but different H^*: $S^1 \vee S^1 \vee S^2$, T^2

There are distinct spaces with *identical* H^*:

$H^*(S^3 \vee S^5) \cong H^*(S(\mathbb{CP}^2)) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(5)}$ degree.

Prop: $\alpha \cup \beta = (-1)^{k+\ell}(\beta \cup \alpha)$ if R commutative.

Künneth Formula

Next goal: $H^*(T^n, \mathbb{Z})$ = free abelian gp with basis

$$\alpha_{i_1} \cup \cdots \cup \alpha_{i_k} \qquad i_1 < \cdots < i_k$$

where $\alpha_i \in H^1(T^n, \mathbb{Z})$ is $p_i^*(\alpha)$ for α a gen of $H^1(S^1, \mathbb{Z})$.
and p_i is projection to i^{th} factor.

Cross Product (aka external cup product)

$$H^*(X, \mathbb{Z}) \times H^*(Y, \mathbb{Z}) \longrightarrow H^*(X \times Y, \mathbb{Z})$$
$$(a, b) \longmapsto p_1^*(a) \cup p_2^*(b)$$

bilinear.

Tensor Products

Bilinear maps are not linear/homomorphisms

e.g. $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(e_1, e_1) \longmapsto 1$

$\Rightarrow (-e_1, -e_1) \longmapsto 1$

$\longrightarrow$ replace $\times$ with $\otimes$

The tensor product of abelian groups A, B is the abelian group $A \otimes B$ with generators $a \otimes b$ $a \in A, b \in B$ and relations

$$(a + a') \otimes b = a \otimes b + a' \otimes b$$
$$a \otimes (b + b') = a \otimes b + a \otimes b'$$

Identity: $0 \otimes 0 = 0 \otimes b = a \otimes 0$

Inverses: $-(a \otimes b) = -a \otimes b = a \otimes -b$.

Universal Property

$$\left\{\begin{matrix}\text{Bilinear maps}\\ \text{from } A\times B\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Homomorphisms}\\ \text{from } A\otimes B\end{matrix}\right\}$$

Basic Properties

(i) $A\otimes B \cong B\otimes A$

(ii) $(\bigoplus A_i)\otimes B \cong \bigoplus (A_i \otimes B)$

(iii) $(A\otimes B)\otimes C \cong A\otimes (B\otimes C)$

(iv) $\mathbb{Z}\otimes A \cong A$

(v) $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$

(vi) $f: A\to A'$, $g: B\to B'$ $\rightsquigarrow$ $f\otimes g : A\otimes B \to A'\otimes B'$

(vii) $\varphi : A\times B \to C$ bilinear $\rightsquigarrow$ $f: A\otimes B\to C$

Back to Cross Product

Property (vii) $\rightsquigarrow$ homomorphism

$$H^*(X,\mathbb{Z})\otimes H^*(Y,\mathbb{Z}) \longrightarrow H^*(X\times Y,\mathbb{Z})$$
$$a\otimes b \longmapsto a\times b$$

The left hand side has multiplication

$$(a\otimes b)(c\otimes d) = (-1)^{|b||c|}\, ac\otimes bd$$

Check: The above map is a ring homomorphism.

THEOREM. (Künneth) $H^*(X,\mathbb{Z}) \otimes H^*(Y,\mathbb{Z}) \xrightarrow{\text{cross}} H^*(X\times Y,\mathbb{Z})$ ring isomorphism
if $H^*(X,\mathbb{Z})$ or $H^*(Y,\mathbb{Z})$ is fin. gen., free.

Exterior Algebras

$\Lambda[\alpha_1, \alpha_2, \ldots]$ = graded tensor product of the $\Lambda[\alpha_i]$, $|\alpha_i|$ odd
As an abelian group, gen by $\alpha_{i_1} \cdots \alpha_{i_k}$ $\quad i_1 < \cdots < i_k$
Multiplication given by $\alpha_i \alpha_j = -\alpha_j \alpha_i \quad i \neq j$
$\Rightarrow \alpha_i^2 = 0.$

Cor: $H^*(T^n, \mathbb{Z}) \cong \Lambda[\alpha_1, \ldots, \alpha_n] \qquad |\alpha_i| = 1.$

$\rightsquigarrow$ elts of H^* are sums of: intersect with oriented coordinate tori

More generally, if X is product of odd-dim spheres
$H^*(X) \cong \Lambda[\alpha_1, \ldots, \alpha_n]$ but $|\alpha_i|$ varies.

For even-dim spheres get $\mathbb{Z}[\alpha]/(\alpha^2)$ factors.

Idea of Proof: Induct on dimension.

Poincaré Duality

For M a compact, orientable n-manifold:

$$H_k(M) \cong H^{n-k}(M)$$

or, modulo torsion:

$$H_k(M) \cong H_{n-k}(M)$$

Examples.
① $H_*(S^n)$ $\quad \mathbb{Z}, 0, \ldots, 0, \mathbb{Z}$
② $H_*(M_g)$ $\quad \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$
③ $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\binom{n}{n-k}} = H_{n-k}(T^n)$

For M a Δ-complex:
compact = finitely many simplices
orientable = $\exists$ choice of $\varepsilon_i \in \{\pm 1\}$ so $\sum_{i=1}^{N} \varepsilon_i \sigma_i$ is a cycle where $\sigma_1, \ldots, \sigma_N$ are the n-simplices of M. The class of such a cycle is called a fundamental class, or orientation. It is written $[M]$.

There are versions of PD for nonorientable manifolds (use $\mathbb{Z}/2\mathbb{Z}$ coefficients) and manifolds with boundary (Lefschetz duality).

One other duality: Alexander duality.
If K is a compact, locally contractible, nonempty proper subspace of S^n, then $\tilde{H}_i(S^n - K) \cong \tilde{H}^{n-i-1}(K)$.

The PD isomorphism will be made explicit:

$$\varphi \mapsto \varphi \frown [M].$$

The Idea of Poincaré Duality: Dual Cell Structures

For manifolds:

cell structures $\leftrightarrow$ dual cell structures

k-cells $\leftrightarrow$ $(n-k)$-cells

$\rightsquigarrow$ face relations reversed.

Examples.
- Platonic solids
- $4g$-gon structure on M_g is self-dual.
- Structure on T^n with one n-cube is self-dual.

Duality with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Can ignore signs $\rightsquigarrow$ There is a natural pairing between a cell structure C and its dual C^*.

$$C_i \leftrightarrow C^*_{n-i}$$

Under this identification $\partial : C_i \longrightarrow C_{i-1}$

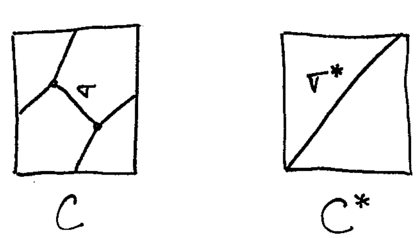
$\sigma \longmapsto$ sum of faces of σ

becomes $\delta : C^*_{n-i} \longrightarrow C^*_{n-i+1}$

$\sigma^* \longmapsto$ sum of dual cells of which σ^* is a face.

$$\rightsquigarrow \underset{\substack{\parallel \\ H_i(M, \mathbb{Z}/2\mathbb{Z})}}{H_i(C, \mathbb{Z}/2\mathbb{Z})} \cong \underset{\substack{\parallel \\ H^{n-i}(M, \mathbb{Z}/2\mathbb{Z})}}{H^{n-i}(C^*, \mathbb{Z}/2\mathbb{Z})}$$

example. T^2



Cap Product

$$\frown : C_k(X) \times C^l(X,\mathbb{Z}) \longrightarrow C_{k-l}(X) \qquad k \geq l$$
$$(\sigma, \varphi) \longmapsto \varphi(\sigma|_{[v_0,\dots,v_l]})\, \sigma|_{[v_l,\dots,v_k]}$$

As usual, need to check this induces a cap product on co/homology. The required formula is:

$$\partial(\sigma \frown \varphi) = (-1)^l (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

$\rightsquigarrow$ cycle $\frown$ cocycle = cycle
cycle $\frown$ coboundary = boundary
boundary $\frown$ cocycle = boundary.

$\rightsquigarrow$ induced cap product

$$H_k(X) \times H^l(X,\mathbb{Z}) \xrightarrow{\frown} H_{k-l}(X)$$

- Linear in each variable
- Natural: $f: X \longrightarrow Y \rightsquigarrow f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$.

<u>Theorem</u> (Poincaré Duality). M = compact n-manifold with orientation $[M]$. Then

$$H^k(M) \longrightarrow H_{n-k}(M)$$
$$\varphi \longmapsto [M] \frown \varphi$$

is an isomorphism.

exercise. check for S^2.

Duality with $\mathbb{Z}$ coefficients

Need to deal with orientations.

Let $M = \Delta$-complex

$[M]$ = orientation

For $\tau = n$-simplex, $\sigma = k$-dim face, define

σ_τ^* = convex hull in τ of barycenters of simplices of τ containing σ

This is $(n-k)$-dim subcomplex of barycentric subdivision $\mathcal{B}(\tau)$.

For $\varphi = k$-cochain, define

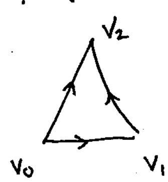
$$D(\varphi) = \sum_{\substack{n\text{-simp } \tau \\ k\text{-simp } \sigma \leq \tau}} \begin{pmatrix} \text{sign of} \\ \tau \text{ in } [M] \end{pmatrix} \begin{pmatrix} \text{sign of} \\ \sigma \text{ in } \tau \end{pmatrix} \varphi(\sigma)\, \sigma_\tau^*$$

~~Simpler way of saying this: orient each simplex of σ_τ^* by embedding in max simplex of $\mathcal{B}(\tau)$, restrict that orientation. Remove this sign term.~~

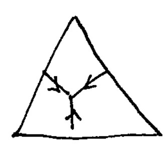
note: simplices of σ_τ^* have orientation induced from canonical orientation of $\mathcal{B}(\tau)$, and sign of σ in τ is ~~whether or not this agrees with the orientation on σ induced by the max. simplex of $\mathcal{B}(\tau)$ containing σ, whose orientation is given by that of τ.~~

defined so σ meets σ_τ^* positively.

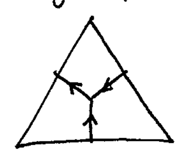
Examples of sign of σ in τ:



τ



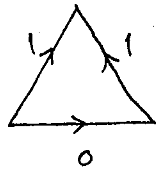
Some 1-cells of $\mathcal{B}(\tau)$ with canon. orient.



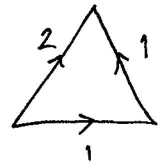
Same 1-cells with coeff = sign of σ in τ

Examples of $D(\varphi)$:

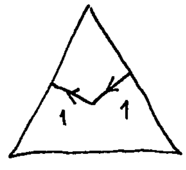
φ_1



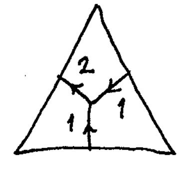
φ_2



$D(\varphi_1)$



$D(\varphi_2)$



THE IDEA OF POINCARÉ DUALITY II

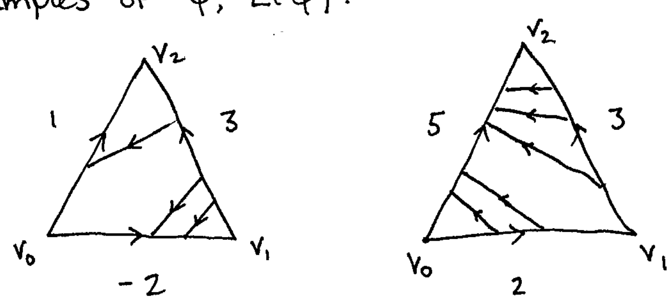
Given φ, want to first relate $D(\varphi)$ and $[M] \cap \varphi$,
then show D is an isomorphism $H^k \to H_{n-k}$.

Restrict to $n=2$, $k=1$.

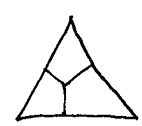
Define an intermediary $L(\varphi)$ = level curves for φ

Claim 1. $L(\varphi)$ is equal to $D(\varphi)$, $[M] \cap \varphi$

Two examples of φ, $L(\varphi)$:



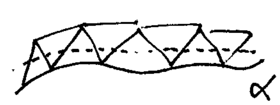
Homotopy $L(\varphi) \rightsquigarrow [M] \cap \varphi$: Push endpts of each edge of $L(\varphi)$ along boundary arrows.

Homotopy $L(\varphi) \rightsquigarrow D(\varphi)$: Push onto 

Claim 2. $L : H^1 \to H_1$ is an isomorphism.

Step 1. φ a coboundary $\Leftrightarrow$ $L(\varphi)$ boundary
$\rightsquigarrow$ L is an injective, well-defined map.

Step 2. L is surjective.

Given cycle α, tile one side by triangles 

Push α up, in general position
$\rightsquigarrow$ the cocycle is intersection with the pushoff.

THE PROOF OF POINCARÉ DUALITY

Cohomology with compact support

Idea: Take cohomology only using cochains φ where, for some compact K, φ kills all chains in $X \setminus K$.

More precisely: $H_c^k(M, R) = \varinjlim_K H^p(X, X \setminus K; R)$

In practice, take the direct limit over some exhaustion.

Example. $H_c^p(\mathbb{R}^n) \cong \mathbb{Z}$

Use exhaustion of $\mathbb{R}^n$ by balls.

LES for cohomology of pairs:

$$0 \longrightarrow H^n(\mathbb{R}^n - B(r)) \xrightarrow{\cong} H^n(\mathbb{R}^n, B(r)) \longrightarrow 0$$

The inclusion $(\mathbb{R}^n, \mathbb{R}^n \setminus B(r+1)) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus B(r))$ clearly induces an $\cong$ on H^n.

Relative cap product

Usual cap product generalizes to

$$H^p(X,A) \times H^q(X,A) \longrightarrow H_{q-p}(X)$$

defined in same way on cochain level.

PD for Noncompact Manifolds

Define $D: H_c^p(M,R) \to H_{n-p}(M,R)$ as the direct limit of maps $D_K: H^p(M, M \setminus K; R) \to H_{n-p}(M,R)$

$$c \mapsto c \frown [M_K]$$

where $[M_K]$ is fundamental class relative to K.

<u>Thm</u>: M = orientable n-manifold

$$D: H_c^p(M,\mathbb{Z}) \to H_{n-p}(M)$$

is an isomorphism.

Steps in the Proof

1. The theorem holds for $M = \mathbb{R}^n$
2. If the theorem holds for $U, V, U \cap V$, it holds for $U \cup V$.
3. If the theorem holds for $U_1 \subseteq U_2 \subseteq \cdots$, it holds for $\bigcup U_i$
4. The theorem holds for open subsets of $\mathbb{R}^n$.
5. The theorem holds for any M.

Steps 1 & 2 are the work. Steps 3-5 are general nonsense.

Step 1. PD holds for $\mathbb{R}^n$.

We saw $H_c^*(\mathbb{R}^n) = \mathbb{Z}_{(n)} = H_{n-*}(\mathbb{R}^n)$

For any K = compact ball, the cap prod. of a generator for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ with $[\mathbb{R}^n_K]$ is $\pm$ the generator for $H_0(\mathbb{R}^n)$ since $\cap$ in this case is evaluation. So the above $\cong$ is indeed induced by $\cap$.

Step 2. PD holds for $U, V, U \cap V \Rightarrow$ PD holds for $U \cup V$.

A Mayer-Vietoris argument.

Step 3. PD holds for $U_1 \subseteq U_2 \subseteq \cdots \Rightarrow$ PD holds for $\cup U_i$

By basic properties of direct limits:

$$H_c^p(\cup U_i) = \varinjlim_i \varinjlim_{K \subset U_i} H^p(U_i, U_i \setminus K) = \varinjlim_i H_c^p(U_i)$$

Also: $$H_{n-p}(\cup U_i) = \varinjlim_i H_{n-p}(U_i)$$

Step 3 follows by naturality of direct limits.

Step 4. PD holds for open subsets of $\mathbb{R}^n$.

Write U as $U_1 \subseteq U_2 \subseteq \cdots$, where U_1 is an open ball, and U_{i+1} obtained from U_i by adding an open ball. B_{i+1}.

Note $B_{i+1} \cap U_i$ is convex, open, has compact closure, so it is homeomorphic to an open ball.

Induction plus Steps 1,2,3.

Step 5. PD holds for any M.

Steps 1 & 4 + Zorn's Lemma $\Rightarrow \exists$ nonempty maximal open set V on which PD holds. If $V \neq M$, can take a coordinate nbhd U disjoint from V.

Steps 1 & 2 $\Rightarrow$ PD holds for $U \cup V$, contradiction.

APPLICATIONS OF POINCARÉ DUALITY

Euler characteristic.

For a manifold M, define

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{rk} H_i(M)$$

<u>Prop</u>: If $\dim(M)$ odd, then $\chi(M) = 0$.
M closed

<u>Prop</u>: If $\dim(M)$ even and $\chi(M)$ odd (e.g. $\mathbb{RP}^2$) then M is not the boundary of any manifold.