Algebraic Topology

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What is algebraic topology?

$$\boxed{\text{Space}} \longrightarrow \boxed{\text{Group}}$$

$X \rightleftarrows \pi_1(X)$ fundamental group

$X \longrightarrow H_k(X)$ k-th homology group

$X \longrightarrow H^k(X)$ k-th cohomology group

What kinds of questions does it answer?

① When are two spaces the same (or not)?

e.g. $\mathbb{R}^m \not\cong \mathbb{R}^n$

what about: $\mathbb{R}^3 -$  vs. $\mathbb{R}^3 -$ 

② Embeddings

What is smallest N s.t. a given manifold embeds in $\mathbb{R}^N$?

Unsolved for $\mathbb{RP}^n$.

③ Fixed point theorems

Brouwer fixed pt theorem: every $D^2 \to D^2$ has a fixed pt.

Borsuk-Ulam theorem.

④ Actions

Which finite groups act freely on S^n?
(known in some cases)
Note: $\mathbb{Z}/n\mathbb{Z} \circlearrowright S^{2k-1}$ $\forall n,k$.

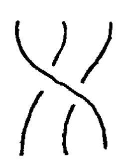
⑤ Sections

What is the largest k s.t. a given manifold admits a continuously varying k-plane field?

Hairy ball theorem.

⑥ Group theory

Every subgroup of a free group is free.
$[F_n, F_n]$ is not finitely generated.
Braid groups are torsion free.



⑦ Algebra

Fundamental theorem of algebra (this week!)

Basic idea of homology

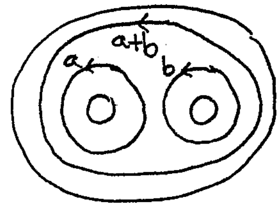
$H_k(X)$ = abelian group of k-dim holes in X

↳ prevents a k-sphere from collapsing

computable

example: X = pair of parts 

$H_1(X) \cong \mathbb{Z}^2$



$H^k(X)$ is dual to $H_k(X)$

$\rightsquigarrow$ consists of functions $H_k(X) \longrightarrow \mathbb{Z}$

Big Goal: Poincaré Duality

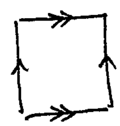
For X = n-manifold $H^k(X) \cong H_{n-k}(X)$

More precisely: the functions in H^k look like "intersect with this fixed element of H_{n-k}"

What do we mean by a space?

Cell complexes aka CW complexes

C = closure finiteness
(closure of open cell hits finitely many open cells)
W = weak topology

e.g. 

Quotient topology: $U \subseteq X/\sim$ is open iff its preimage in X is open.

We build CW complexes inductively

(i) Start with a discrete set of points X^0. The points are regarded as 0-cells.

(ii) Inductively form n-skeleton X^n from X^{n-1} by attaching n-cells D^n_α via
$$\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$$

X^n has quotient topology.

(iii) Either stop at a finite stage, or continue indefinitely.

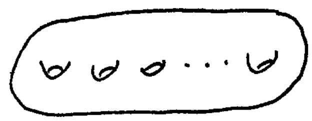
In latter case, use weak topology: a set is open iff its intersection with each cell is open.

$\dim(X) = \sup$ of dim of cells

Examples of CW Complexes

① 1-dim CW complexes are graphs.

② $(4g+2)$-gon with opposite sides identified



③ $S^n = e^0 \cup e^n$ $\qquad e^i = i$-cell.

④ $\mathbb{RP}^n$ = space of lines in $\mathbb{R}^{n+1}$

$= e^0 \cup e^1 \cup \cdots \cup e^n$

To see this: $\mathbb{RP}^n = S^n / \text{antipodal map}$

$= D^n / \text{antipodal map on } \partial D^n = S^{n-1}$

So on ∂D^n see $\mathbb{RP}^{n-1}$, and we glue D^n to that.

⑤ $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^n$ $\qquad$ exercise.

Subcomplexes

Subcomplex = closed ~~subsp~~ union of cells.

A subcomplex of a CW complex is a CW complex.

example: k-skeleton.

Equivalence of Spaces

Intuition: Two spaces are equivalent if one can be deformed into the other

Special case: A <u>deformation</u> <u>retraction</u> $X \to A$ is a continuous family

$$\{ f_t : X \to X \mid t \in I \}$$

s.t.
$$f_0 = id$$
$$f_1(X) = A$$
$$f_t|_A = id \quad \forall t.$$

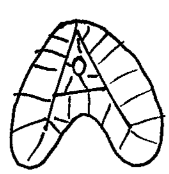
Continuous means
$$X \times I \to X$$
$$(x,t) \mapsto f_t(x)$$

is continuous.

Example: Given $f : X \to Y$, the <u>mapping</u> <u>cylinder</u> is

$$M_f = (X \times I) \amalg Y \,/ \sim$$

where $(x,1) \sim f(x)$

e.g. 

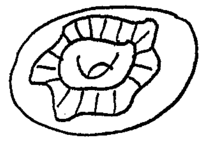
X = boundary
Y = core

Fact: M_f deformation retracts to Y.

Homotopy Equivalence

A *homotopy* is a continuous family
$$\{ f_t : X \to Y \mid t \in I \}$$

examples: deformation retraction



A map $f: X \to Y$ is a *homotopy equivalence* if there is a $g: Y \to X$ such that
$$fg \simeq id \quad \text{and} \quad gf \simeq id$$
homotopic

Say: X & Y are homotopy equivalent, or have the same homotopy type. $\quad X \simeq Y$

Exercise: This is an equivalence relation.

Fact: If A is a deformation retract of X, then $X \simeq A$

Exercise:  all homotopy equiv.

Exercise: $\mathbb{R}^n \simeq *$ $\quad$ Say $\mathbb{R}^n$ is contractible.

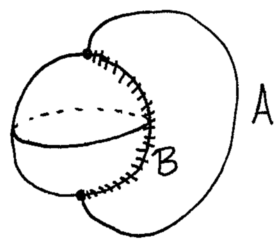
Read: House with 2 rooms, Hatcher p. 4.

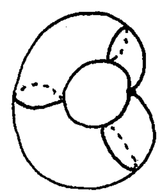
Two Criteria for Homotopy Equivalence

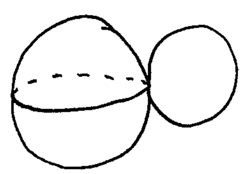
① (X, A) = CW-pair (i.e. A subcomplex of X)
A contractible
$\Longrightarrow X \simeq X/A$ ← identify A to one point

example: X = graph
A = any edge connecting distinct vertices.

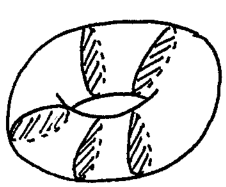
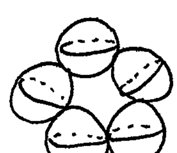
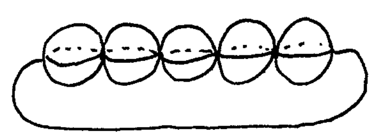
Thus any graph $\simeq$  = wedge of circles $S^1 \vee \cdots \vee S^1$

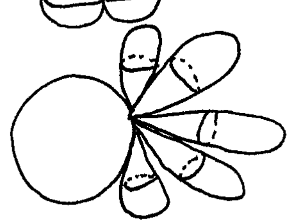
example: $X =$ 

$X/A \cong$  $\cong S^2 / \text{north pole} \sim \text{south pole}$

$X/B \cong$  $= S^2 \vee S^1$

$X \simeq X/A \simeq X/B$

exercise:  $\simeq$  $\cong$ 

$\simeq$ 

② (X,A) CW-pair
$f, g : A \to Y$ homotopic (i.e. $\exists$ homotopy f_t, $f_0 = f$, $f_1 = g$)
$\Longrightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

Note: $X \sqcup_f Y = (X \amalg Y)/a \sim f(a)$

exercise: Do last example using Criterion ②

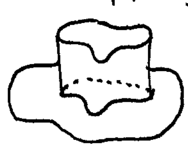
Proofs of both criteria use Homotopy Extension Property.

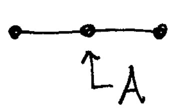
Say a pair of spaces (X,A) has the homotopy extension property if whenever we have

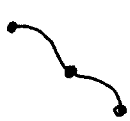
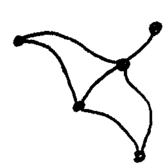
$$f_0 : X \to Y$$
$$f_t : A \to Y \quad \text{homotopy}$$

we can extend f_t to X.

In other words every map $M_i \to Y$ can be extended to $X \times I \to Y$
where M_i = mapping ~~cyl~~ cylinder of $i : A \to X$ inclusion.



example. $X =$  $Y = \mathbb{R}^2$

$f_0 =$  $f_t =$  extension: 

A <u>retraction</u> of a space X onto a subspace A is

$$r: X \to A$$
$$\text{s.t. } r|_A = id.$$

<u>Prop</u>: (X,A) has HEP $\Longleftrightarrow$ M_i is a retract of $X \times I$ where $i: A \longrightarrow X$ inclusion.

<u>Proof</u>: $\Longrightarrow$ Set $Y = M_i$, $f_0 = id$.

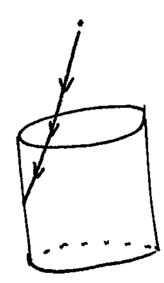
$\Longleftarrow$ $X \times I \xrightarrow{r} M_i \xrightarrow{f_t} Y$

Note: f_t deformation retract of X to A
$\Rightarrow f_1 : X \to A$ a retraction of X to A

<u>Prop</u>: If $(X,A) = CW$ pair, then M_i is a deformation retract of $X \times I$ (where $i: A \to X$ incl.)

In particular, (X,A) has HEP.

<u>Proof</u>: First do $X = D^n$ $A = \partial D^n$ via projection:



Retract each n-cell of $X^n - A^n$ during $[\frac{1}{2}^{n+1}, \frac{1}{2}^n]$

Continuous since it is on each cell (no problem near 0 since each n-skeleton stationary in $[0, \frac{1}{2}^{n+1}]$).

Prop: (X,A) has HEP
A contractible
$\Rightarrow q : X \to X/A$ is a homotopy equivalence

Idea: Need inverse to q. Contract A, extend to $f_t : X \to X$.
Since $f_1(A) = \text{pt.}$ can regard $f_1 : X/A \to X$.

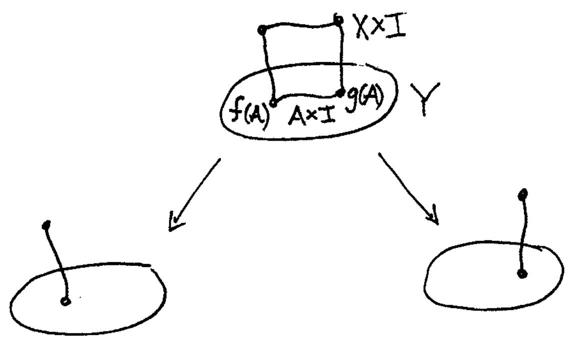
exercise: read/write details.

example. $X = \mathbb{R}$ $A = [-1,1]$

Prop: $(X,A) = $ CW pair
$f,g : A \to Y$ homotopic
$\Rightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

Idea: Show both are deformation retractions of
$$(X \times I) \sqcup_F Y$$
where $F : A \times I \to Y$ is homotopy from f to g.

example: $X = $  A $\quad Y = D^2$



exercise: write details

note: use existence of deformation retraction $X \times I \to M_i$
(stronger than HEP).