Applications of Homology

① Jordan Curve Theorem, etc.

homeo onto image. in this case, any injective continuous map.

<u>Theorem</u>. Let $h: S^1 \to \mathbb{R}^2$ embedding.
Then $\mathbb{R}^2 - h(S^1)$ has exactly 2 connected components.

Easy for nice curves (e.g. polygonal). Must consider things like Osgood curves, which have positive (exterior) area (these are obtained by perturbing space filling curves).

<u>Prop</u>: (a) If $h: D^k \to S^n$ an embedding, then
$$\tilde{H}_i(S^n - h(D^k)) = 0 \quad \forall i$$
(b) If $h: S^k \to S^n$ an embedding, $k < n$, then
$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise} \end{cases}$$

(b) implies any S^{n-1} in S^n divides S^n into two components, each with homology of a point.
For $n=2$, Jordan Curve Thm.
For $n=3$, it is possible for one component to be not simply connected. (Alexander horned sphere.)

(b) also implies $H_1(S^3 - \text{knot}) \cong \mathbb{Z}$.

Proof of Prop: (a) Induct on k

$k=0 \rightsquigarrow S^n - h(D^k) \cong \mathbb{R}^n$ ✓

Replace D^k with I^k.

Let $A = S^n - h(I^{k-1} \times [0, 1/2])$

$B = S^n - h(I^{k-1} \times [1/2, 1])$

Induction $\Rightarrow \tilde{H}_i(A \cup B) = \tilde{H}_i(S^n - h(I^{k-1} \times 1/2)) = 0$.

Mayer-Vietoris $\Rightarrow$

$$\Phi : \underset{S^n - h(D^k)}{\underset{\parallel}{\tilde{H}_i(A \cap B)}} \longrightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B) \quad \text{isomorphism } \forall i.$$

So if $[\alpha] \neq 0$ in $\tilde{H}_i(S^n - h(D^k))$ then

$\alpha \neq 0$ in $H_i(S^n -$ half of $h(I^k))$

Say these halves converge to $I^{k-1} \times \{p\}$.

By above, α a boundary in $\tilde{H}_i(S^n - h(I^{k-1} \times \{p\}))$

Say $\alpha = \partial \beta$.

β compact $\Rightarrow [\alpha] = 0$ at some finite stage.

$\rightsquigarrow$ contradiction. ▨

(b) Induct on k.

$k=0 \rightsquigarrow S^n - h(S^0) \simeq S^{n-1} \times \mathbb{R}$ ✓

Let $S^k = D^k_+ \cup_{S^{k-1}} D^k_-$

$A = S^n - h(D^k_+)$, $B = S^n - h(D^k_-)$

Mayer-Vietoris plus (a) $\Rightarrow$

$$\tilde{H}_{i+1}(S^n - h(S^{k-1})) \cong \tilde{H}_i(S^n - h(S^k))$$ ▨

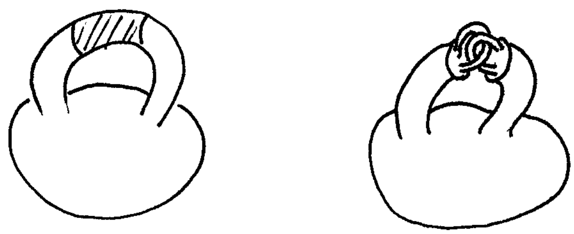
Exercise. Examine the case $k=n$

$\rightsquigarrow S^n$ cannot embed in $\mathbb{R}^n$

$\mathbb{R}^m$ cannot embed in $\mathbb{R}^n$ $m > n$.

Aside: Alexander Horned Sphere

The Alexander Horned Ball is the intersection $\bigcap_{i=1}^{\infty} X_i$



X_1 X_2

$$\pi_1(AHB^c) = \langle \alpha_0, \alpha_1, \dots \mid [\alpha_1, \alpha_2] = \alpha_0,\ [\alpha_3, \alpha_4] = \alpha_1,\ [\alpha_5, \alpha_6] = \alpha_2,\ \dots \rangle$$

This group is nontrivial – it is an increasing union of free groups. But since each α_i is a commutator, the abelianization is trivial.

② Invariance of Domain

<u>Theorem</u> U open in $\mathbb{R}^n$, $h: U \to \mathbb{R}^n$ embedding
$\Rightarrow h(U)$ open in $\mathbb{R}^n$.

<u>Proof</u> Think of $\mathbb{R}^n$ as S^n - pt.
Equivalent to show $h(U)$ open in S^n.
Let $x \in U$, D^n = disk about x in U.
Suffices to show $h(\operatorname{int} D^n)$ open in S^n
Prop (b) $\Rightarrow S^n - h(D^n)$ has 2 path components.
The components are $h(\operatorname{int} D^n)$, $S^n - h(D^n)$. Indeed:
· Since $h(\operatorname{int} D^n)$ path conn, these sets are disjoint
· $S^n - h(D^n)$ path conn by Prop (b)
Since $S^n - h(\partial D^n)$ open in S^n ($h(\partial D^n)$ compact in Hausdorff),
its path components = connected components (true for loc. comp.)
An open set with finitely many comp. must have
each comp. open
$\Rightarrow h(\operatorname{int} D^n)$ open in $S^n - h(\partial D^n)$
$\Rightarrow$ open in S^n ▨

<u>Cor</u>: M = compact n-manifold, N = connected n-manifold
Then any embedding $M \xrightarrow{h} N$ is surjective, hence a homeo.

<u>Proof</u>: $h(M)$ closed in N (compact in Hausdorff)
Since N conn, suffices to show $h(M)$ open in N.
Let $x \in M$. Choose neighborhood V of $h(x)$ homeo to $\mathbb{R}^n$.
Choose nbhd U of x in $h^{-1}(V)$ homeo to $\mathbb{R}^n$
$h|_U$ an embedding into V. Thm $\Rightarrow h(U)$ open in V,
hence open in N. ▨

③ Division Algebras

An <u>algebra</u> over $\mathbb{R}$ is $\mathbb{R}^n$ with bilinear multiplication

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(a, b) \mapsto ab$$

So: $a(b+c) = ab+ac$, $(a+b)c = ac+bc$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$

It is a <u>division algebra</u> if $ax=b$, $xa=b$ always solvable for $a \neq 0$. ("no zero divisors")

Four classical examples: $\mathbb{R}$, $\mathbb{C}$, Quaternions, Octonians

<u>Theorem</u>. $\mathbb{R}$ & $\mathbb{C}$ are the only finite dimensional division algebras over $\mathbb{R}$ that are commutative and have id.

<u>Proof</u>. We'll show: a fin. dim. comm. div alg. has dim ≤ 2.
Suppose $\mathbb{R}^n$ has a comm. div. alg. structure.
Define $f: S^{n-1} \to S^{n-1}$ by $f(x) = x^2/|x^2|$
$\leadsto$ induced map $\bar{f}: \mathbb{R}P^{n-1} \to S^{n-1}$
Claim: $\bar{f}$ injective

Pf: $f(x) = f(y) \Rightarrow x^2 = \alpha^2 y^2 \Rightarrow x^2 - \alpha^2 y^2 = 0$
$\Rightarrow (x+\alpha y)(x-\alpha y) = 0$ (commutativity)
No zero div $\Rightarrow x = \pm \alpha y \Rightarrow x = \pm y$.

$\bar{f}$ injective map of compact Hausdorff $\Rightarrow$ embedding
Cor $\Rightarrow \bar{f}$ surjective if $n > 1$.
$\Rightarrow \mathbb{R}P^{n-1} \cong S^{n-1} \Rightarrow n=2$ (compare π_1)

A little more algebra to get full theorem. ▨

DEGREE

$$f : S^n \to S^n \rightsquigarrow f_* : H_n(S^n) \to H_n(S^n)$$
$$\alpha \mapsto d\alpha$$

d = <u>degree</u> of f.

<u>Facts</u> (i) $\deg id = 1$
(ii) $\deg f = 0$ if f not surjective
(iii) $\deg f = \deg g \Leftrightarrow f \simeq g$ $\Rightarrow$ due to Hopf.
(iv) $\deg fg = \deg f \deg g$
(v) $\deg f = -1$ f = reflection along equator
(vi) $\deg(\text{antipodal}) = (-1)^n$

④ Hairy Ball Theorem

<u>Theorem</u>. S^n has a continuous field of nonzero tangent vectors iff n is odd.

<u>Proof</u>. ($\Rightarrow$) Let $v(x)$ = vector field on S^n. Translate $v(x)$ to origin
$\rightsquigarrow v(x) \perp x$ in $\mathbb{R}^{n+1}$
$v(x) \neq 0 \ \forall x \rightsquigarrow$ can replace $v(x)$ with $v(x)/|v(x)|$
$\Rightarrow (\cos t)x + (\sin t)v(x)$ = unit S^1 in x-$v(x)$ plane
$f_t(x) = (\cos t)x + (\sin t)v(x)$ a homotopy from id ($t=0$) to antipodal map ($t=\pi$)
(iii) $\Rightarrow \deg id = \deg antip.$
(i), (vi) $\Rightarrow n$ odd.

($\Leftarrow$) For $n = 2k-1$ set $v(x_1, \dots, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$. ▨

One more fact about degree:

(vi) If f has no fixed points, then $\deg f = (-1)^{n+1}$

proof: find homotopy to antipodal map (straight line)

⑤ Prop: $\mathbb{Z}/2\mathbb{Z}$ is only group that can act freely on S^n if n is even.

Pf: Say $G \curvearrowright S^n \rightsquigarrow d: G \to \{\pm 1\}$ homomorphism by (iv)

Action free $\Rightarrow d(g) = (-1)^{n+1}$ $g \neq id$ by (vi)

n even $\Rightarrow \ker d = 1 \Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$. ▨

Can also use degree to define/compute cellular homology

$\rightsquigarrow$ compute homology of $\mathbb{CP}^n$, $S^n \times S^n$, T^n, $\mathbb{RP}^n$, $L(p,q)$, etc.

see text.

⑥ Borsuk-Ulam Theorem

Prop: Say $f: S^n \to S^n$, $f(-x) = -f(x)$ $\forall x$ (odd map).
Then f has odd degree.

Theorem: $g: S^n \to \mathbb{R}^n \Rightarrow \exists\, x$ s.t. $g(x) = g(-x)$.

Proof: Let $f(x) = g(x) - g(-x)$, say $f(x) \neq 0$ $\forall x$.

Replace $f(x)$ by $f(x)/|f(x)|$

$\rightsquigarrow f: S^n \to S^{n-1}$ odd

Prop $\Rightarrow f|_{\text{equator}}$ has odd degree.

But either hemisphere gives a nullhomotopy.

CONTRADICTION. ▨

⑦ Lefschetz Fixed Point Theorem

Trace: for $\varphi : A \to A$ A = f.g. abelian group
$$tr\,\varphi = tr(A/\text{torsion} \longrightarrow A/\text{torsion})$$

X = space with finitely generated homology, trivial H_i $i \geq N$.
e.g. finite simplicial complex.

The <u>Lefschetz number</u> of $f : X \longrightarrow X$ is
$$\tau(f) = \sum (-1)^i \, tr(f_* : H_i(X) \longrightarrow H_i(X))$$

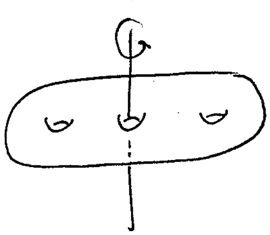
<u>Theorem</u> $\tau(f)$ = sum of indices of fixed points (assume fixed points are isolated)

In particular $\tau(f) \neq 0 \Rightarrow$ fixed points
Brouwer FPT is corollary.

~~Recall~~ Index of fixed point p is $\deg(\bar{f} : (X, X-p) \longrightarrow (X, X-p))$

<u>Linear maps</u>. Modulo torsion, $\mathbb{RP}^n$ n even has homology of pt.
$\Rightarrow$ every map has a fixed point
$\Rightarrow$ every linear map $\mathbb{R}^n \to \mathbb{R}^n$, n odd
has an eigenvector (can also use elementary reasoning).

Can do many examples of LFPT with surfaces, e.g.

 or 

Preparation: Approximation by simplicial maps

Simplicial maps. K, L simplicial complexes
$K \to L$ simplicial if simplices $\mapsto$ simplices, linearly.

<u>Theorem</u>. K = finite simplicial complex, L = simplicial complex.
Any $f: K \to L$ is homotopic to a map that is simplicial w.r.t. some subdivision of K.

<u>Idea of Proof</u> that $\tau(f) \neq 0 \Rightarrow \exists$ fixed points.

Assume $f: X \to X$ has no fixed points
Simplicial approx $\rightsquigarrow g: X \to X$ simplicial, homotopic to f
$g(\sigma) \cap \sigma = \emptyset \;\; \forall$ simplices σ.

Note $\tau(f) = \tau(g)$.
To show $tr(g_*) = 0$ in all dim.

Key: $\tau(g) = \sum (-1)^n \, tr(g_*: H_n(X^n, X^{n-1}) \to H_n(X^n, X^{n-1}))$
Use the fact that g takes X^n to X^n plus some algebra.

Since g crushes/permutes cells without fixing any, all of these traces are 0. $\square$