Cohomology

Same basic information as homology, but get
  
  - multiplicative structure
  - pairing with homology
  - contravariance

Quick idea:  \( X = \Delta \)-complex
\[ G \] = abelian group, say \( \mathbb{Z} \)
\( \Delta^i(X) = \) functions from \( i \)-simplices of \( X \) to \( G \).
\[ \Delta^i(X) \rightarrow \mathbb{Z} \]
\( \delta \) : \( \Delta^i(X,G) \rightarrow \Delta^{i+1}(X,G) \) coboundary \[ (-1)^i f(\partial_i \sigma) \]

\[ \delta f(\sigma) = \sum_{\sigma = \tau + e} (-1)^i f(\sigma) \]

\( H^*(X;G) \) is homology of this chain complex

Graphs. \( X = 1 \)-dim \( \Delta \)-complex = oriented graph

Let \( f \in \Delta^0(X,G) \)
\[ \delta f(e) = f(v_e) - f(v_0) \]

= change of \( f \) over \( e \)  "derivative"

think: \( f \) = elevation

\[ \rightarrow \text{chain complex:} \]
\[ 0 \rightarrow \Delta^0(X,G) \rightarrow \Delta^1(X,G) \rightarrow 0 \]

\[ H^0(X,G) = \ker \delta \]

= functions constant on each component
= direct product of components

(as opposed to direct sum in homology case)
\[ H'(X, G) = \Delta'(X, G) / \text{Im} \delta \]

So for \( f \in \Delta'(X, G) \), have \( [f] = 0 \) in \( H'(X, G) \) iff \( f \) has an antiderivative.

**Examples.**

1. \( X = \text{tree} \)
   Antiderivatives always exist  
   \[ \Rightarrow H'(X, G) = 0. \]

2. \( X = \emptyset \)
   \[ \Delta'(X, G) \cong G \]
   No nontrivial function has an antiderivative  
   \[ \Rightarrow H'(X, G) \cong G \]

3. \( X = \bigvee_{\alpha} S^1 \)
   \[ \Rightarrow H'(X, G) \cong \prod_{\alpha} G \]

**More generally.** \( X = \text{any tree graph} \).

Let \( T = \text{maximal tree (or forest)} \), \( E = \text{edges outside } T \)
\[ \Rightarrow H'(X, G) = \bigoplus_{E} G \]
(again, instead of direct sum).

Why? First consider \( \{ f \mid f|_T = 0 \} \)
Two of these are cohomologous \( \iff \) they are equal  
(only possible antiderivative is \( F = \text{const} \)).

Next show any \( f' \in \Delta' \) is cohomologous to some \( f \) with \( f|_T = 0 \). Modify \( f' \) by making one edge of \( T \) evaluate to \( 0 \), say add \( g \) to \( f'(e) \).
Then for any edge \( e \) of \( X - T \), either add or subtract \( g \), depending on whether loop through \( e, e' \) traverses them in same or diff directions.
Check new \( f' \) cohomologous to old.
Two dimensions. \( X = 2 \)-dim \( \Delta \)-complex

\[ \delta : \Delta'(X,G) \rightarrow \Delta^2(X,G) \]

\[ \delta f([v_0,v_1,v_2]) = f([v_1,v_2]) - f([v_0,v_2]) + f([v_0,v_1]) \]

Check that

\[ 0 \rightarrow \Delta^0(X,G) \rightarrow \Delta'(X,G) \rightarrow \Delta^2(X,G) \rightarrow 0 \]

is a chain complex: say \( f \in \Delta^0(X,G) \).

\[ \delta \delta f([v_0,v_1,v_2]) = (f(v_2) - f(v_0)) + (f(v_1) - f(v_0)) - (f(v_2) - f(v_1)) \]

i.e. if you hike a loop, total elevation change is zero.

1-cocycles: \( \delta f = 0 \) iff

\[ f([v_0,v_2]) = f([v_0,v_1]) + f([v_1,v_2]) \]

so \( \delta f \) measures failure of additivity.

This is the local obstruction to \( f \) being in \( \text{im} \delta \)

And \( f \neq 0 \) in \( H^1(X) \) \( \iff \) does not come from \( F \in \Delta^0 \).

i.e. if there is a global obstruction.

Analogue with calculus. 1-forms on \( \mathbb{R}^3 \leftrightarrow \) vector fields

Want to know if vector field is \( \nabla f \)

local obstruction: \text{curl} = 0. \quad \text{(closed)}

global obstruction: line integrals = 0. \quad \text{(exact)}

In \( \mathbb{R}^n \), all closed forms are exact.

Not true in other spaces, e.g. \( \mathbb{R}^2 - \{0\} \)

de Rham cohomology: closed forms / exact forms.
Geometric interpretation of 1-cocycles, X a surface.

Take $G = \mathbb{Z}_2$. $\delta f = 0$ means $f$ takes value 1 on even # of edges in each $\Delta$.

$\{ f \} = 0 \iff$ can color regions black & white.

Examples: disk, annulus:

\[\begin{array}{c}
\text{unlabeled } = 0.
\end{array}\]

Take $G = \mathbb{Z}$. Again $\delta f = 0 \quad \rightarrow \text{ collection of curves}$

\[\begin{array}{c}
e.g. \quad \text{or} \quad \text{arrows point up.}
\end{array}\]

$[f] = 0 \iff$ can assign elevation to each vertex consistently.

Exercise: Construct nontrivial cocycle on annulus.

So: in annulus, can walk in a loop and change your elevation!

cf. international dateline.

Exercise: Find geometric interpretations of 1- & 2-cocycles in a 3-manifold.
Cohomology Groups (Some Abstract Algebra)

Start with a chain complex of abelian groups $C$:

$$
\cdots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots
$$

$$
\Rightarrow H_n(C) = \ker \partial_n / \text{im } \partial_{n+1}
$$

To get cohomology, we dualize: replace each $C_n$ with its dual $C_n^* = \text{Hom}(C_n, G)$

replace each $\partial$ with $\delta = \partial^* : C_{n-1}^* \rightarrow C_n^*$

Notice: $\delta \delta = \partial^* \partial^* = (\partial \partial^*)^* = 0^* = 0$

$$
\Rightarrow H^n(C, G) = \ker \delta / \text{im } \delta
$$

Guess: $H^n(C, G) \cong \text{Hom}(H^n(C), G)$ too optimistic, but almost true.

It is true for graphs.

Example.

$$
C : 0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow 0
$$

$$
\Rightarrow H_0(C) = \mathbb{Z}, \; H_1(C) = \mathbb{Z}/2\mathbb{Z}, \; H_2(C) = 0, \; H_3(C) = \mathbb{Z}
$$

$C^* : 0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0$

$$
\Rightarrow H^0(C, \mathbb{Z}) = \mathbb{Z}, \; H^1(C, \mathbb{Z}) = 0, \; H^2(C, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \; H^3(C, \mathbb{Z}) = \mathbb{Z}
$$

See: Torsion shifts up one dimension.

This holds in general, since any chain complex of finitely generated free abelian groups splits as a direct sum of

$$
0 \rightarrow \mathbb{Z} \rightarrow 0
$$

and

$$
0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow 0
$$
Universal Coefficient Theorem for Cohomology

\[ C : \cdots \to C_n \to C_{n-1} \to H_n(C) \]

\[ T_n(C) = \text{torsion subgroup of } H_n(C). \]

We just showed: If the \( H_n(C) \) are finitely generated, and each \( C_i \) is free abelian, then

\[ H^n(C, \Z) \cong H_n(C)/T_n(C) \oplus T_{n-1}(C) \]

This is a special case of:

Theorem. There is a split short exact sequence:

\[ 0 \to \text{Ext}(H_{n-1}(C), G) \to H^n(C, G) \to \text{Hom}(H_n(C), G) \to 0 \]

The group \( \text{Ext}(H_{n-1}(C), G) \) is explicit. It describes all extensions of \( H_{n-1}(C) \) by \( G \). Some properties: If \( H \) is finitely gen, then

1. \( \text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G) \)
2. \( \text{Ext}(H, G) = 0 \) if \( H \) is free
3. \( \text{Ext}(\Z/n\Z, G) \cong G/nG \)

These imply the special case of UCT above.

Universal coefficient theorem for homology:

\[ H_n(X, \Q) \cong H_n(X, \Z) \otimes \Q \quad \text{ (later).} \]
**Cohomology of Spaces**

\( \chi = \text{space}, \ G = \text{abelian group} \)

\( C^n(\chi, G) = \text{singular } n\text{-chains with coefficients in } G, \text{ except allow } \infty \text{ sums} \)

\( = \text{dual of } C_n(\chi) \)

\( = \text{Hom}(C_n(\chi), G) \)

Coboundary \( \delta \) is \( \partial \): for \( \varphi \in C^n(\chi, G) \)

\[ \delta \varphi : C^{n+1}(\chi) \xrightarrow{\partial} C^n(\chi) \xrightarrow{\varphi} G. \]

Again, \( \delta^2 = 0 \).

\[ \sim H^n(\chi, G) \quad \text{cohomology group with coefficients in } G. \]

\[ = \ker \delta / \text{im } \delta = \text{cocycles/coboundaries} \]

Cocycles. A cochain \( \varphi \) is a cocycle iff \( \delta \varphi = \varphi \partial = 0 \),

i.e. \( \varphi \) vanishes on all boundaries.

It is a coboundary if it has an "antiderivative."

Since \( C_n(\chi) \) free, UCT gives:

\[ 0 \rightarrow \text{Ext}(H_{n-1}(\chi), G) \rightarrow H^n(\chi, G) \rightarrow \text{Hom}(H_n(\chi), G) \rightarrow 0 \]

"Cohomology groups of \( \chi \) with arbitrary coefficients is determined by the homology groups of \( \chi \) with \( \mathbb{Z} \) coefficients."

What is \( \text{Ext} \)?

Let \( B_n = \text{im } \partial_{n+1} \) (boundaries)

\( Z_n = \ker \partial_n \) (cycles)

\[ \sim \text{ in } : B_n \rightarrow Z_n \]

\[ \text{Ext } (H_{n-1}(\chi), G) = \text{Coker } i_{n-1}^* \]

\( \sim \text{dual to } i_{n-1} \)
COHOMOLOGY IN LOW DIMENSIONS

\( n = 0 \) \quad \text{Ext term is trivial, so}
\[ H^n(X, G) \cong \operatorname{Hom}(H^0(X), G) \]

Can see directly from definitions:
- Sing. 0-simplices \( \leftrightarrow \) points of \( X \)
- Cochains \( \leftrightarrow \) functions \( X \to G \) (not continuous)
- Cocycles \( \leftrightarrow \) vanish on boundaries
  \( \leftrightarrow \) const. on each path component
\[ \Rightarrow H^n(X, G) = \text{functions } \{ \text{path components of } X \} \to G \]
\[ \cong \text{Hom}(H_0(X), G) \].

\( n = 1 \) \quad \text{Ext = 0 since } H_0(X) \text{ free}
\[ \Rightarrow H^1(X, G) \cong \text{Hom}(H_1(X), G) \]
\[ \cong \text{Hom}(\pi_1(X), G) \text{ if } X \text{ path conn.} \]

COEFFICIENTS IN A FIELD

\( H_n(X, F) = \) homology gps of chain complex of \( F \)-vector spaces \( C_n(X, F) \)
Dual complex \( \text{Hom}_F(C_n(X, F), F) = \text{Hom}(C_n(X), F) \)
\[ \cong H^n(X, F) \]
Can generalize UCT to fields (or pid's) \( \Rightarrow \text{Ext vanishes for fields} \)
\[ \cong H^n(X, F) \cong \text{Hom}_F(H_0(X, F), F) \]

For \( F = \mathbb{Z}/p\mathbb{Z} \text{ or } \mathbb{Q}, \text{ Hom}_F = \text{Hom} \)
Examples of $2$-cocycles

1. $X = D^2$

   We know $H^2(D^2, \mathbb{Z}) = 0$

   So $\varphi = \delta \psi$.

   What is $\psi$?

   Solution:

   No obstructions.

2. $X = S^2$

   Want to show $[\varphi] \neq 0$

   in $H^2(S^2, \mathbb{Z})$

   i.e. no antiderivative $\psi$.

   Any $\psi$ with $\delta \psi = \varphi$ must satisfy:

   writing $a$ for $\psi(a)$

   \[
   \begin{align*}
   b + d &= a \\
   e + c &= a \\
   b + f &= c \\
   e + f &= d + 1
   \end{align*}
   \]

   \[
   \Rightarrow (b+d) - (e+c) = 1 \\
   \Rightarrow a - a = 1.
   \]

3. $X = T^3$, $G = \mathbb{Z}/2\mathbb{Z}$.

   Realize $T^3$ as $\Delta$-complex by subdividing cube

   into 6 tetrahedra, identifying opp faces of the

   cube. Let $L$ = line segment in cube that is a loop

   in $T^3$, misses 1-skeleton. Declare $\varphi(T) = 1$ if

   $T \cap L \neq \emptyset$. Show $[\varphi] \neq 0$ in $H^2(T^3, \mathbb{Z}/2\mathbb{Z})$. 

Cohomology Theory

Reduced groups, relative groups, long exact seq of pair, excision, Mayer-Vietoris, all work for cohomology.

Induced Homomorphisms - Contravariance

Given \( f: X \to Y \), get chain maps \( f^*: C_*(X) \to C_*(Y) \).
Dualize: \( f^*: C^n(Y) \to C^n(X) \)
\( f^* \delta = \delta f^* \) dualizes to \( \delta f^* = f^* \delta \)
\( \implies f^*: H^n(Y, G) \to H^n(X, G) \)
with: \( (fg)^* = g^* f^* \) \& \( (id)^* = id \)
Say \( X \to H^n(X, G) \) is a contravariant functor.

Homotopy Invariance

\( f \simeq g : X \to Y \implies f^* = g^* : H^n(Y) \to H^n(X) \).
Dualize the proof for homology
Recall there is a chain homotopy \( P \) s.t. \( g^* - f^* = \delta P + P \delta \)
Dualize: \( g^* - f^* = P^* \delta + \delta P^* \)
\( \implies P^* \) a chain homotopy between \( f^* \) & \( g^* \)
So all the work has been done.