Cohomology

Same basic information as homology, but get

- multiplicative structure
- pairing with homology
- contravariance

Quick idea: $X = \Delta$-complex

G = abelian group, say $\mathbb{Z}$

$\Delta^i(X)$ = functions from i-simplices of X to G.

= homomorphisms $\Delta_i(X) \to G$

$\delta : \Delta^i(X,G) \to \Delta^{i+1}(X,G)$ coboundary

For $f \in \Delta^i$, σ an $\cancel{i}$ $(i+1)$-simplex, $\delta f(\sigma) = \sum (-1)^k f(\partial_k \sigma)$

$H^*(X;G)$ is homology of this chain complex

~~Example~~ Graphs. X = 1-dim Δ-complex = oriented graph

Let $f \in \Delta^0(X,G)$

$\delta f(e) = f(v_t) - f(v_0)$

= change of f over e "derivative"

think: f = elevation

$\rightsquigarrow$ chain complex:

$$0 \to \Delta^0(X,G) \xrightarrow{\delta} \Delta^1(X,G) \to 0$$

$H^0(X,G) = \ker \delta$

= functions constant on each component

= direct product of components

(as opposed to direct sum in homology case)

$H^1(X,G) = \Delta^1(X,G)/\mathrm{Im}\,\delta$

So for $f \in \Delta^1(X,G)$, have $[f]=0$ in $H^1(X,G)$ iff f has an antiderivative.

Examples. ① X = tree

Antiderivatives always exist

$\Rightarrow H^1(X,G) = 0$.

② $X = \circlearrowright$

$\Delta^1(X,G) \cong G$

No nontrivial function has an antiderivative

$\rightsquigarrow H^1(X,G) \cong G$

③ $X = \bigvee_\alpha S^1$

$\rightsquigarrow H^1(X,G) \cong \prod_\alpha G$

More generally. X = any ~~tree~~ graph.

Let T = maximal tree (or forest), E = edges outside T

$\rightsquigarrow H^1(X,G) = \prod_E G$ (again, instead of direct sum).

Why? First consider $\{f \mid f|_T = 0\}$

Two of these are cohomologous $\Leftrightarrow$ they are equal (only possible antiderivative is F = const).

Next show any $f' \in \Delta^1$ is cohomologous to some f with $f|_T = 0$. Modify f' by making one edge e of T evaluate to 0, say add g to $f'(e)$. Then for any edge e' of $X-T$, either add or subtract g, depending on whether loop through e,e' traverses them in same or diff directions. Check new f' cohomologous to old.

Two dimensions. $X = 2$-dim Δ-complex

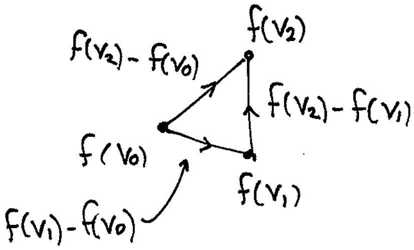
$$\delta : \Delta^1(X,G) \to \Delta^2(X,G)$$

$$\delta f([v_0,v_1,v_2]) = f([v_1,v_2]) - f([v_0,v_2]) + f([v_0,v_1])$$

Check that

$$0 \to \Delta^0(X,G) \to \Delta^1(X,G) \to \Delta^2(X,G) \to 0$$

is a chain complex: say $f \in \Delta^0(X,G)$.



$$\delta\delta f([v_0,v_1,v_2]) = (f(v_1) - f(v_0)) + (f(v_2) - f(v_1)) - (f(v_2) - f(v_0))$$

i.e. if you hike a loop, total elevation change is zero.

1-cocycles: $\delta f = 0$ iff

$$f([v_0,v_2]) = f([v_0,v_1]) + f([v_1,v_2])$$

so δf measures failure of additivity.

This is the <u>local</u> obstruction to f being in $\operatorname{im} \delta$

And $f \neq 0$ in $H^1(X) \iff$ does not come from $F \in \Delta^0$.

i.e. if there is a <u>global</u> obstruction.

Analogue with calculus. 1-forms on $\mathbb{R}^3 \longleftrightarrow$ vector fields

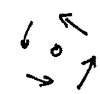
Want to know if vector field is ∇f

local obstruction: curl $= 0$. (closed)

global obstruction: line integrals $= 0$. (exact)

In $\mathbb{R}^n$, all closed forms are exact.

Not true in other spaces, e.g. $\mathbb{R}^2 - \{0\}$



de Rham cohomology: closed forms / exact forms.

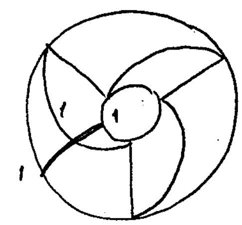
Geometric interpretation of 1-cocycles, X a surface.

Take $G = \mathbb{Z}_2$. $\delta f = 0$ means f takes value 1 on even # of edges in each Δ.

$\rightsquigarrow$ collection of curves, arcs

$[f] = 0 \iff$ can color regions black & white.

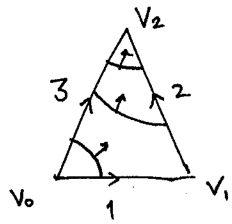
examples. disk, annulus:



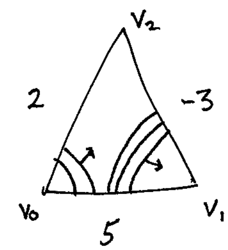
unlabeled $= 0$.

Take $G = \mathbb{Z}$. Again $\delta f = 0 \longrightarrow$ collection of curves

e.g.



or



arrows point up.

$[f] = 0 \iff$ can assign elevation to each vertex consistently.

exercise. Construct nontrivial cocycle on annulus.

So: in annulus, can walk in a loop and change your elevation!

cf. international dateline.

Exercise: Find geometric interpretations of 1- & 2-cocycles in a 3-manifold.

Cohomology Groups (Some Abstract Algebra)

Start with a chain complex of abelian groups C:

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

$$\rightsquigarrow H_n(C) = \ker\partial_n / \mathrm{im}\,\partial_{n+1}$$

To get cohomology, we dualize: replace each C_n with its dual

$$C_n^* = \mathrm{Hom}(C_n, G)$$

replace each ∂ with $\delta = \partial^* : C_{n-1}^* \to C_n^*$

Notice: $\delta\delta = \partial^*\partial^* = (\partial\partial)^* = 0^* = 0.$

$$\rightsquigarrow H^n(C,G) = \ker\delta / \mathrm{im}\,\delta$$

Guess: $H^n(C,G) \cong \mathrm{Hom}(H^n(C), G)$ Too optimistic, but almost true. It is true for graphs.

Example.

$$\begin{array}{ccccccccccc} & & C_3 & & C_2 & & C_1 & & C_0 & & \\ C: & 0 \to & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to & 0 \end{array}$$

$\rightsquigarrow H_0(C) = \mathbb{Z}$, $H_1(C) = \mathbb{Z}/2\mathbb{Z}$ $H_2(C) = 0$ $H_3(C) = \mathbb{Z}$

use formula $\delta\varphi = \varphi\partial$

$$C^*: \quad 0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

$\rightsquigarrow H^0(C,\mathbb{Z}) = \mathbb{Z}$, $H^1(C,\mathbb{Z}) = 0$, $H^2(C,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $H^3(C,\mathbb{Z}) = \mathbb{Z}$

See: Torsion shifts up one dimension.

This holds in general, since any chain complex of finitely generated free abelian groups splits as a direct sum of $0 \to \mathbb{Z} \to 0$

and $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$

Universal Coefficient Theorem for Cohomology

$C: \quad \cdots \to C_n \to C_{n-1}$ chain complex.

$\rightsquigarrow H_n(C)$

$T_n(C)$ = torsion subgroup of $H_n(C)$.

We just showed: If the $H_n(C)$ are finitely generated, and each C_i is free abelian, then

$$H^n(C, \mathbb{Z}) \cong H_n(C)/T_n(C) \oplus T_{n-1}(C)$$

This is a special case of:

<u>Theorem</u>. There is a split short exact sequence:

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C, G) \to Hom(H_n(C), G) \to 0$$

The group $Ext(H_{n-1}(C), G)$ is explicit. It describes all extensions of $H_{n-1}(C)$ by G. Some properties: If H is finitely gen, then

① $Ext(H \oplus H', G) \cong Ext(H, G) \oplus Ext(H', G)$

② $Ext(H, G) = 0$ if H is free

③ $Ext(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

These imply the special case of UCT above.

Universal coefficient theorem for homology:

$$H_n(X, \mathbb{Q}) \cong H_n(X, \mathbb{Z}) \otimes \mathbb{Q} \qquad \text{(later).}$$

Cohomology of Spaces

X = space, G = abelian group

$C^n(X,G)$ (= singular n-chains with coefficients in G, except allow ∞ sums)

= dual of $C_n(X)$

= $\mathrm{Hom}(C_n(X), G)$

Coboundary δ is ∂_*: for $\varphi \in C^n(X,G)$

$$\delta\varphi : C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G.$$

Again, $\delta^2 = 0$.

$\rightsquigarrow$ $H^n(X,G)$ cohomology group with coefficients in G.

$$= \ker\delta / \operatorname{im}\delta = \text{cocycles}/\text{coboundaries}$$

Cocycles. A cochain φ is a cocycle iff $\delta\varphi = \varphi\partial = 0$,

i.e. φ vanishes on all boundaries.

It is a coboundary if it has an "antiderivative."

Since $C_n(X)$ free, UCT gives:

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X,G) \to \mathrm{Hom}(H_n(X), G) \to 0$$

"Cohomology groups of X with arbitrary coefficients is determined by the homology groups of X with $\mathbb{Z}$ coefficients."

What is Ext?

Let $B_n = \operatorname{im}\partial_{n+1}$ (boundaries)

$Z_n = \ker\partial_n$ (cycles)

$\rightsquigarrow$ $i_n : B_n \to Z_n$

$\mathrm{Ext}(H_{n-1}(X), G) = \mathrm{Coker}\, i_{n-1}^*$

dual to i_{n-1}

Cohomology in Low Dimensions

$\underline{n=0}$ Ext term is trivial, so

$$H^0(X,G) \cong \mathrm{Hom}(H_0(X),G)$$

Can see directly from definitions:

sing. 0-simplices $\longleftrightarrow$ points of X

cochains $\longleftrightarrow$ functions $X \to G$ (not continuous)

cocycles $\longleftrightarrow$ vanish on boundaries

$\longleftrightarrow$ const. on each path component

$\Rightarrow H^0(X,G) =$ functions $\{$path components of $X\} \to G$

$= \mathrm{Hom}(H_0(X),G)$.

$\underline{n=1}$ Ext $= 0$ since $H_0(X)$ free

$\Rightarrow H^1(X,G) \cong \mathrm{Hom}(H_1(X),G)$

$\cong \mathrm{Hom}(\pi_1(X),G)$ if X path conn.

Coefficients in a Field

$H_n(X,F) =$ homology gps of chain complex of F-vector spaces $C_n(X,F)$

Dual complex $\mathrm{Hom}_F(C_n(X,F),F) = \mathrm{Hom}(C_n(X),F)$

$\rightsquigarrow H^n(X,F)$

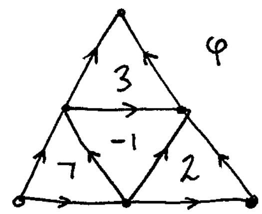
Can generalize UCT to fields (or pid's) $\rightsquigarrow$ Ext vanishes for fields

$\rightsquigarrow H^n(X,F) \cong \mathrm{Hom}_F(H_n(X,F),F)$

For $F = \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$, $\mathrm{Hom}_F = \mathrm{Hom}$

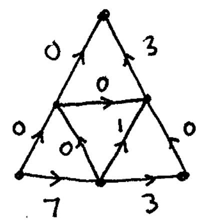
Examples of 2-cocycles

① $X = D^2$



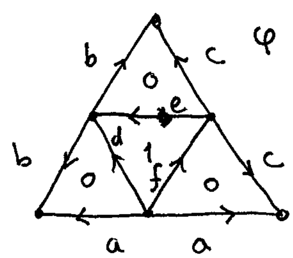
We know $H^2(D^2, \mathbb{Z}) = 0$
so $\varphi = \delta\psi$.
What is ψ?

Solution:



No obstructions.

② $X = S^2$



Want to show $[\varphi] \neq 0$
in $H^2(S^2, \mathbb{Z})$
i.e. no antiderivative ψ.

Any ψ with $\partial\psi = \varphi$ must satisfy:

writing
a for $\psi(a)$

$$b + d = a$$
$$e + c = a$$
$$\left.\begin{array}{l} b + f = c \\ e + f = d + 1 \end{array}\right\} \Rightarrow (b+d) - (e+c) = 1$$
$$\Rightarrow a - a = 1.$$

③ $X = T^3$, $G = \mathbb{Z}/2\mathbb{Z}$.

Realize T^3 as Δ-complex by subdividing cube into 6 tetrahedra, identifying opp faces of the cube. Let L = line segment in cube that is a loop in T^3, misses 1-skeletion. Declare $\varphi(T) = 1$ if $T \cap L \neq \phi$. Show $[\varphi] \neq 0$ in $H^2(T^3, \mathbb{Z}/2\mathbb{Z})$.

Cohomology Theory

Reduced groups, relative groups, long exact seq of pair, excision, Mayer-Vietoris, all work for cohomology.

Induced Homomorphisms - Contravariance

Given $f: X \to Y$, get chain maps $f_\#: C_n(X) \to C_n(Y)$

Dualize: $f^\#: C^n(Y,G) \to C^n(X,G)$

$f_\# \partial = \partial f_\#$ dualizes to $\delta f^\# = f^\# \delta$

$\rightsquigarrow f^*: H^n(Y,G) \to H^n(X,G)$

with: $(fg)^* = g^* f^*$ & $(id)^* = id$

Say $X \mapsto H^n(X,G)$ is a contravariant functor.

Homotopy Invariance

$f \simeq g: X \to Y \Rightarrow f^* = g^*: H^n(Y) \to H^n(X)$.

Dualize the proof for homology

Recall there is a chain homotopy P s.t. $g_\# - f_\# = \partial P + P\partial$

Dualize: $g^\# - f^\# = P^* \delta + \delta P^*$

$\rightsquigarrow P^*$ a chain homotopy between $f^\#$ & $g^\#$

So all the work has been done.