Poincaré Duality

For M a compact, orientable n-manifold:

$$H_k(M) \cong H^{n-k}(M)$$

or, modulo torsion:

$$H_k(M) \cong H_{n-k}(M)$$

Examples. ① $H_*(S^n)$ $\quad \mathbb{Z}, 0, \ldots, 0, \mathbb{Z}$

② $H_*(M_g)$ $\quad \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$

③ $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\binom{n}{n-k}} = H_{n-k}(T^n)$

For M a Δ-complex:

compact = finitely many simplices

orientable = $\exists$ choice of $\varepsilon_i \in \{\pm 1\}$ so $\sum_{i=1}^{N} \varepsilon_i \sigma_i$ is a cycle where $\sigma_1, \ldots, \sigma_N$ are the n-simplices of M. The class of such a cycle is called a fundamental class, or orientation. It is written $[M]$.

There are versions of PD for nonorientable manifolds (use $\mathbb{Z}/2\mathbb{Z}$ coefficients) and manifolds with boundary (Lefschetz duality).

One other duality: Alexander duality.

If K is a compact, locally contractible, nonempty proper subspace of S^n, then $\tilde{H}_i(S^n - K) \cong \tilde{H}^{n-i-1}(K)$.

The PD isomorphism will be made explicit:

$$\varphi \mapsto \varphi \frown [M].$$

The Idea of Poincaré Duality: Dual Cell Structures

For manifolds:

cell structures $\leftrightarrow$ dual cell structures

k-cells $\leftrightarrow$ $(n-k)$-cells

$\rightsquigarrow$ face relations reversed.

Examples.
- Platonic solids
- $4g$-gon structure on M_g is self-dual.
- Structure on T^n with one n-cube is self-dual.

Duality with $\mathbb{Z}/2\mathbb{Z}$ coefficients

Can ignore signs $\rightsquigarrow$ There is a natural pairing between a cell structure C and its dual C^*.

$$C_i \leftrightarrow C^*_{n-i}$$

Under this identification $\partial : C_i \longrightarrow C_{i-1}$

$\sigma \longmapsto$ sum of faces of σ

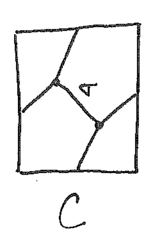
becomes $\delta : C^*_{n-i} \longrightarrow C^*_{n-i+1}$

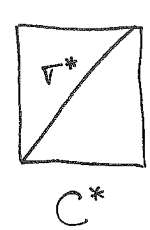
$\sigma^* \longmapsto$ sum of dual cells of which σ^* is a face.

$$\rightsquigarrow H_i(C, \mathbb{Z}/2\mathbb{Z}) \cong H^{n-i}(C^*, \mathbb{Z}/2\mathbb{Z})$$

$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ H_i(M, \mathbb{Z}/2\mathbb{Z}) & & H^{n-i}(M, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

example. T^2





Cap Product

$$\frown : C_k(X) \times C^\ell(X,\mathbb{Z}) \longrightarrow C_{k-l}(X) \qquad k \geq l$$
$$(\sigma, \varphi) \longmapsto \varphi(\sigma|_{[v_0,\dots,v_l]})\, \sigma|_{[v_l,\dots,v_k]}$$

As usual, need to check this induces a cap product on co/homology. The required formula is:

$$\partial(\sigma \frown \varphi) = (-1)^l(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

$\rightsquigarrow$ cycle $\frown$ cocycle = cycle

cycle $\frown$ coboundary = boundary

boundary $\frown$ cocycle = boundary.

$\rightsquigarrow$ induced cap product

$$H_k(X) \times H^\ell(X,\mathbb{Z}) \xrightarrow{\frown} H_{k-\ell}(X)$$

- Linear in each variable
- Natural: $f : X \longrightarrow Y \rightsquigarrow f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$.

<u>Theorem</u> (Poincaré Duality). M = compact n-manifold with orientation $[M]$. Then

$$H^k(M) \longrightarrow H_{n-k}(M)$$
$$\varphi \longmapsto [M] \frown \varphi$$

is an isomorphism.

exercise. check for S^2.

Duality with $\mathbb{Z}$ coefficients

Need to deal with orientations.

Let $M = \Delta$-complex

$[M]$ = orientation

For $\tau = n$-simplex, $\sigma = k$-dim face, define

σ^*_τ = convex hull in τ of barycenters of simplices of τ containing σ

This is $(n-k)$-dim subcomplex of barycentric subdivision $\mathcal{B}(\tau)$.

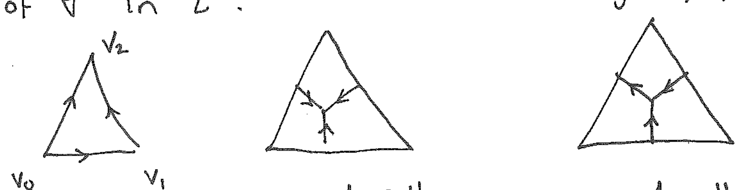
For $\varphi = k$-cochain, define

$$D(\varphi) = \sum_{\substack{n\text{-simp } \tau \\ k\text{-simp } \sigma \leq \tau}} \begin{pmatrix} \text{sign of} \\ \tau \text{ in } [M] \end{pmatrix} \begin{pmatrix} \text{sign of} \\ \sigma \text{ in } \tau \end{pmatrix} \varphi(\sigma)\, \sigma^*_\tau$$

~~Simpler way of saying this: orient each simplex of σ^*_τ by embedding in max simplex of $\mathcal{B}(\tau)$, restrict that orientation. Remove this sign term.~~

note: simplices of σ^*_τ have orientation induced from canonical orientation of $\mathcal{B}(\tau)$, and sign of σ in τ is ~~whether or not this agrees with the orientation on σ induced by the max. simplex of $\mathcal{B}(\tau)$ containing σ, whose orientation is given by that of τ.~~ defined so σ meets σ^*_τ positively.

Examples of sign of σ in τ:

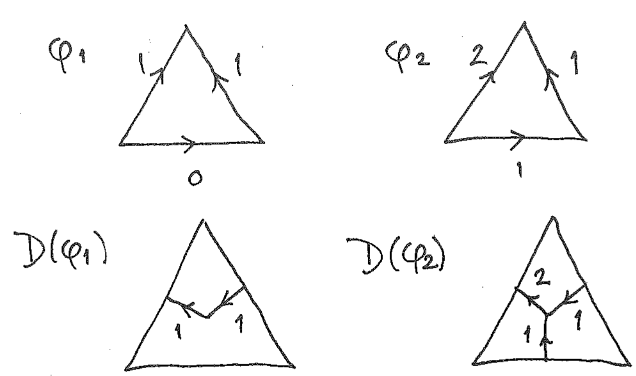


τ

Some 1-cells of $\mathcal{B}(\tau)$ with canon. orient.

Same 1-cells with coeff = sign of σ in τ

Examples of $D(\varphi)$:



The Idea of Poincaré Duality II

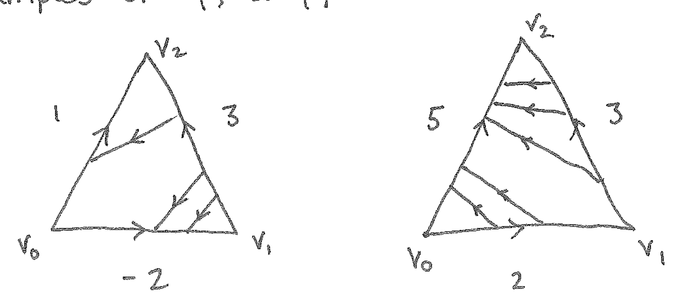
Given φ, want to first relate $D(\varphi)$ and $[M] \cap \varphi$,
then show D is an isomorphism $H^k \to H_{n-k}$.

Restrict to $n=2$, $k=1$.

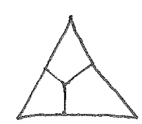
Define an intermediary $L(\varphi)$ = level curves for φ

Claim 1. $L(\varphi)$ is equal to $D(\varphi)$, $[M] \cap \varphi$

Two examples of φ, $L(\varphi)$:

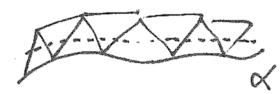


Homotopy $L(\varphi) \rightsquigarrow [M] \cap \varphi$: Push endpts of each edge of $L(\varphi)$ along boundary arrows.

Homotopy $L(\varphi) \rightsquigarrow D(\varphi)$: Push onto 

Claim 2. $L : H^1 \to H_1$ is an isomorphism.

Step 1. φ a coboundary $\Leftrightarrow$ $L(\varphi)$ boundary
$\rightsquigarrow$ L is an injective, well-defined map.

Step 2. L is surjective.
Given cycle α, tile one side by triangles 

Push α up, in general position
$\rightsquigarrow$ the cocycle is intersection with the pushoff.

THE PROOF OF POINCARÉ DUALITY

Cohomology with compact support

Idea: Take cohomology only using cochains φ where, for some compact K, φ kills all chains in $X \setminus K$.

More precisely: $H_c^k(M,R) = \varinjlim_K H^p(X, X\setminus K; R)$

In practice, take the direct limit over some exhaustion.

Example. $H_c^p(\mathbb{R}^n) \cong \mathbb{Z}$

Use exhaustion of $\mathbb{R}^n$ by balls.

LES for cohomology of pairs:

$$0 \longrightarrow H^n(\mathbb{R}^n - B(r)) \xrightarrow{\cong} H^n(\mathbb{R}^n, B(r)) \longrightarrow 0$$

The inclusion $(\mathbb{R}^n, \mathbb{R}^n \setminus B(r+1)) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus B(r))$ clearly induces an $\cong$ on H^n.

Relative cap product

Usual cap product generalizes to

$$H^p(X,A) \times H^q(X,A) \longrightarrow H_{q-p}(X)$$

defined in same way on cochain level.

PD for Noncompact Manifolds

Define $D: H^p_c(M,R) \longrightarrow H_{n-p}(M,R)$ as the direct limit of maps $D_K: H^p(M, M\setminus K; R) \longrightarrow H_{n-p}(M,R)$

$$c \longmapsto c \frown [M_K]$$

where $[M_K]$ is fundamental class relative to K.

<u>Thm</u>: M = orientable n-manifold

$$D: H^p_c(M,\mathbb{Z}) \longrightarrow H_{n-p}(M)$$

is an isomorphism.

Steps in the Proof

1. The theorem holds for $M = \mathbb{R}^n$
2. If the theorem holds for $U, V, U \cap V$, it holds for $U \cup V$.
3. If the theorem holds for $U_1 \subseteq U_2 \subseteq \cdots$, it holds for $\bigcup U_i$
4. The theorem holds for open subsets of $\mathbb{R}^n$.
5. The theorem holds for any M.

Steps 1 & 2 are the work. Steps 3-5 are general nonsense.

Step 1. PD holds for $\mathbb{R}^n$.

We saw $H_c^*(\mathbb{R}^n) = \mathbb{Z}_{(n)} = H_{n-*}(\mathbb{R}^n)$
For any K = compact ball, the cap prod. of a generator for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ with $[\mathbb{R}^n_K]$ is $\pm$ the generator for $H_0(\mathbb{R}^n)$ since $\frown$ in this case is evaluation. So the above $\cong$ is indeed induced by $\frown$.

Step 2. PD holds for $U, V, U \cap V \Rightarrow$ PD holds for $U \cup V$.

A Mayer-Vietoris argument.

Step 3. PD holds for $U_1 \subseteq U_2 \subseteq \cdots \Rightarrow$ PD holds for $\bigcup U_i$

By basic properties of direct limits:

$$H_c^p(\bigcup U_i) = \varinjlim_i \varinjlim_{K \subset U_i} H^p(U_i, U_i \setminus K) = \varinjlim_i H_c^p(U_i)$$

Also:
$$H_{n-p}(\bigcup U_i) = \varinjlim_i H_{n-p}(U_i)$$

Step 3 follows by naturality of direct limits.

Step 4. PD holds for open subsets of $\mathbb{R}^n$.

Write U as $U_1 \subseteq U_2 \subseteq \cdots$, where U_1 is an open ball, and U_{i+1} obtained from U_i by adding an open ball. B_{i+1}.
Note $B_{i+1} \cap U_i$ is convex, open, has compact closure, so it is homeomorphic to an open ball.
Induction plus Steps 1, 2, 3.

Step 5. PD holds for any M.

Steps 1 & 4 + Zorn's Lemma $\Rightarrow \exists$ nonempty maximal open set V on which PD holds. If $V \neq M$, can take a coordinate nbhd U disjoint from V.
Steps 1 & 2 $\Rightarrow$ PD holds for $U \cup V$, contradiction.

APPLICATIONS OF POINCARÉ DUALITY

Euler characteristic.

For a manifold M, define

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{rk} H_i(M)$$

Prop: If $\dim(M)$ odd, then $\chi(M) = 0$. (M closed)

Prop: If $\dim(M)$ even and $\chi(M)$ odd (e.g. $\mathbb{R}P^2$) then M is not the boundary of any manifold.