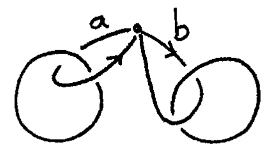
Fundamental Group

$\pi_1(X)$ = group of homotopy classes of based paths in X.

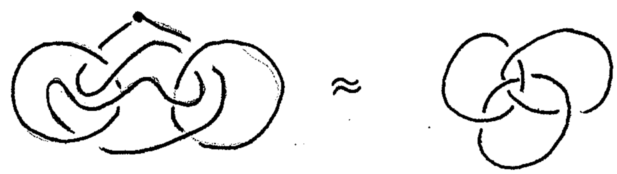
Will see: $X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$

Examples: ① $\mathbb{R}^3$ - unknot $\rightsquigarrow \mathbb{Z}$

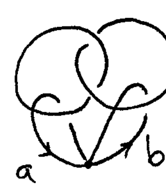
② $\mathbb{R}^3$ - unlink



$aba^{-1}b^{-1}$:



③ $\mathbb{R}^3$ - Hopf link



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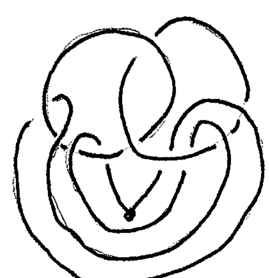


= id

Is π_1 abelian?

push these two strands in tandem around the left-hand circle to see triviality.

Formal Definitions

A <u>path</u> in a space X is a map $I \to X$

A <u>homotopy of paths</u> is a homotopy $f_t : I \to X$ such that $f_t(0)$ and $f_t(1)$ are independent of t.

example. Any two paths f_0, f_1 in $\mathbb{R}^n$ with same endpoints are homotopic via straight-line homotopy:

$$f_t(s) = (1-t)\, f_0(s) + t f_1(s)$$

exercise. Homotopy of paths is an equivalence relation. $\simeq$

The <u>composition</u> of paths f, g with $f(1) = g(0)$ is the path

$$fg(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

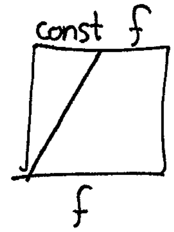
exercise. $f_0 \simeq f_1$, $g_0 \simeq g_1 \Rightarrow f_0 g_0 \simeq f_1 g_1$

A <u>loop</u> is a path f with $f(0) = f(1)$.

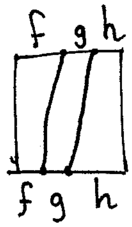
The fundamental group of X (based at x_0) is the group of homotopy classes of loops based at x_0 under composition. Write $\pi_1(X, x_0)$.

Prop: $\pi_1(X, x_0)$ is a group.

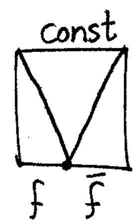
Proof: Identity = constant loop



Associativity:



Inverses:



$\bar{f}(t) = f(1-t)$

Prop: X = path connected, $x_0, x_1 \in X$
$\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1)$

The isomorphism is not canonical!

Say X is simply connected if ① X is path connected
② $\pi_1(X) = 1$.

This terminology is explained by:

Prop: X is simply connected $\Leftrightarrow$ *there is a unique homotopy class of paths joining any two points of X.*

Fact: Contractible $\Rightarrow$ simply connected.

FUNDAMENTAL GROUP OF THE CIRCLE

<u>Thm</u>: $\pi_1(S^1) \cong \mathbb{Z}$

<u>Proof outline</u>: Given a loop $f: I \to S^1$, want to find a <u>lift</u>, that is:

$$\tilde{f}: I \to \mathbb{R}$$

such that $\tilde{f}(0) = 0$, $p\tilde{f} = f$ ← ignore the international date line.

The map $\pi_1(S^1) \to \mathbb{Z}$ is

$$f \mapsto \tilde{f}(1)$$

Well-definedness: existence/uniqueness of lifts
Multiplicativity: easy
Injectivity: homotopic loops have homotopic lifts
Surjectivity: easy

Remains to show loops lift uniquely and homotopies lift.

Idea: Cover S^1 by small pieces whose preimages in $\mathbb{R}$ are unions of open intervals.
Given a loop/homotopy, cut it into pieces, lift piece by piece.

Proof thus follows from Lemma below.

<u>Lemma</u>: Given $F : Y \times I \to S^1$
$\tilde{F} : Y \times \{0\} \to \mathbb{R}$ lift of $F|_{Y\times\{0\}}$
$\exists!\ \tilde{F} : Y \times I \to \mathbb{R}$ lifting F, extending $\tilde{F}|_{Y\times\{0\}}$.

Path lifting: $Y = \{y_0\}$ Homotopy lifting: $Y = I$.

<u>Proof</u> ($Y = \{y_0\}$ case): Write I for $y_0 \times I$.

Cover S^1 by $\{U_\alpha\}$ so that $\forall \alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α.

F continuous $\Rightarrow$ can choose
$$0 = t_0 < t_1 < \cdots < t_m = 1$$
so that $\forall i$, $F([t_i, t_{i+1}])$ is contained in some U_α; call it U_i.

Say $\tilde{F}$ defined on $[0, t_i]$, $\tilde{F}(t_i) \in \tilde{U}_i$,
$p|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ homeo.

Define $\tilde{F}$ on $[t_i, t_{i+1}]$ via
$$(p|_{\tilde{U}_i})^{-1} \circ F|_{[t_i, t_{i+1}]}$$

Induct. ▨

Exercise. Prove for general Y.

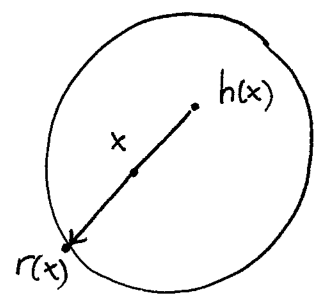
Prop: $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$
for X, Y path connected.

Cor: $\pi_1(T^2) \cong \mathbb{Z}^2$

APPLICATIONS

Brouwer Fixed Point Theorem: *Every $h: D^2 \to D^2$ has a fixed point.*

Proof: Say $h(x) \neq x \;\; \forall \; x \in D^2$.
Can define $r: D^2 \to S^1$ via
retraction



Let f_0 = loop in $S^1 = \partial D^2$
f_t = any homotopy to a point in D^2
$\Rightarrow r f_t$ = homotopy in S^1 of f_0 to trivial loop.
Thus $\pi_1(S^1) = 1$. Contradiction ▨

Also:

Borsuk-Ulam theorem – for any $f: S^2 \to \mathbb{R}^2$, $\exists$ antipodal pair $x, -x$ s.t. $f(x) = f(-x)$.

Ham Sandwich theorem.

Thm: If we write S^2 as a union of 3 closed sets, at least one must contain a pair of antipodal points.

Fundamental Theorem of Algebra: *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof: Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$

Define $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$,

$\pi : \mathbb{C} - 0 \longrightarrow S^1$

$\alpha \longmapsto \alpha/|\alpha|$,

$R > |a_1| + \cdots + |a_n| + 1$,

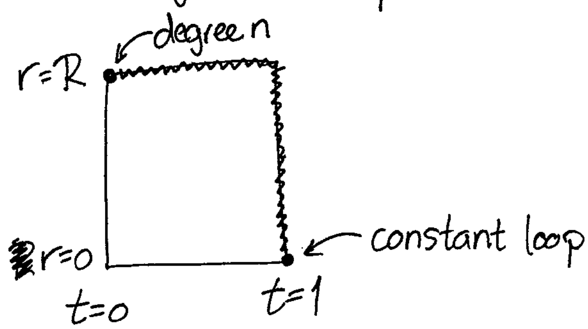
$f_{r,t}(s) : S^1 \to S^1$

$f_{r,t}(s) = \pi \circ p_t(re^{2\pi i s})$

Claim: p_t has no roots on $|z| = R$ for $t \in I$.

$\Rightarrow f_{R,t}(s)$ defined.

Thus the shaded path gives a homotopy from constant loop in S^1 to degree n loop $\Rightarrow n = 0$.



Proof of Claim: For $|z| = R$,

$$|z^n| = R^n = R \cdot R^{n-1} > (|a_1| + \cdots + |a_n|)\,|z^{n-1}|$$

$$\geq |a_1 z^{n-1} + \cdots + a_n|$$

(But $|\alpha| > |\beta| \Rightarrow \alpha + \beta \neq 0$.) ∎

Induced Homomorphisms

$$\varphi : (X, x_0) \longrightarrow (Y, y_0)$$
$$\rightsquigarrow \varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[f] \longmapsto [\varphi f]$$

Functoriality ① $(\varphi\psi)_* = \varphi_* \psi_*$
② $id_* = id$

<u>Fact</u>: φ a homeomorphism $\Rightarrow$ φ_* an isomorphism
<u>Proof</u>: $\varphi_* \varphi^{-1}_* = (\varphi\varphi^{-1})_* = id_* = id$

<u>Prop</u>: $\pi_1(S^n) = 1$ for $n \geq 2$.
<u>Proof</u>: $S^n - pt \cong \mathbb{R}^n$, which is contractible.
By Fact, suffices to show any loop in S^n is homotopic to one that is not surjective.

<u>Prop</u>: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, $n > 2$.
<u>Proof</u>: $\mathbb{R}^n - pt \cong S^{n-1} \times \mathbb{R}$

$$\pi_1(S^{n-1} \times \mathbb{R}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \cong \begin{cases} \mathbb{Z} & n=2 \\ 1 & n>2 \end{cases}$$

Apply Fact.

Prop: If $\varphi: X \to Y$ homotopy equivalence, then $\varphi_*: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ isomorphism.

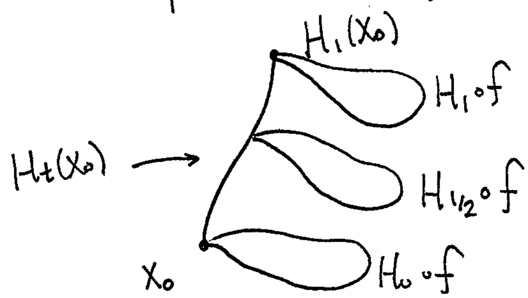
Proof: Let $\psi: Y \to X$ homotopy inverse.
So $\varphi\psi \simeq id$.

Remains to show: $H_t : X \to X$ homotopy
$H_0 = id$
$\Rightarrow (H_1)_* : \pi_1(X, x_0) \to \pi_1(X, H_1(x_0))$
an isomorphism.

We already know the path $H_t(x_0)$ gives

$$\pi_1(X, x_0) \xrightarrow{\cong} \pi_1(X, H_1(x_0))$$
$$f \longmapsto \overline{H_t(x_0)}\, f\, H_t(x_0)$$

But latter path $\simeq H_1 \circ f = (H_1)_*(f)$



So $(H_1)_*$ an isomorphism. $\boxtimes$

Prop: $i : A \to X$ inclusion.

X retracts to $A \Rightarrow i_*$ injective

X deformation retracts to $A \Rightarrow i_*$ isomorphism.

exercise. T^2 retracts to S^1.

In group theory, a <u>retraction</u> is a homomorphism $\rho : G \to H$, where $H < G$, with $\rho|_H = id$.

$\Rightarrow G \cong H \rtimes \ker\rho$.

FREE GROUPS AND FREE PRODUCTS

$F_n = \{\text{reduced words in } x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$

multiplication: concatenate, reduce. associativity nontrivial!

$G * H = \{\text{reduced words in } G, H\}$

$*_\alpha G_\alpha$ similar $= \{g_1 \cdots g_m \quad g_i \in G_{\alpha_i} \quad \alpha_i \neq \alpha_{i+1} \quad g_i \neq id\}$

example. Infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

= symmetries of 

Properties

1. $G_\alpha \leq * G_\alpha$
2. $\bigcap G_\alpha = 1$
3. Any collection $G_\alpha \to H$ extends uniquely to $* G_\alpha \to H$