Van Kampen's Theorem

$X = A \cup B$ A, B open, path connected.
$A \cap B$ path connected.

$x_0 \in A \cap B$ basepoint for $X, A, B, A \cap B$.

The induced $\pi_1(A) \to \pi_1(X)$ & $\pi_1(B) \to \pi_1(X)$ extend to

$$\Phi : \pi_1(A) * \pi_1(B) \to \pi_1(X)$$

Denote $i_A : A \to X$, $i_B : B \to X$.

Let N = normal subgroup of $\pi_1(A) * \pi_1(B)$ generated by the $i_A(\omega)\, i_B(\omega)^{-1}$ for $\omega \in \pi_1(A \cap B)$.

<u>Theorem</u>: ① Φ is surjective
② $\ker \Phi = N$.

Examples. ① $\pi_1(S^1 \vee S^1) \cong F_2$

induction $\rightsquigarrow$ $\pi_1(\bigvee_n S^1) \cong F_n$

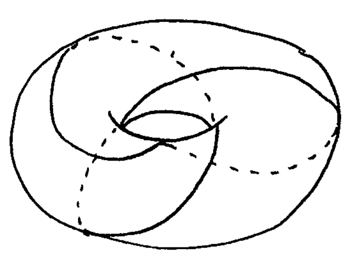
$\Rightarrow \pi_1(\mathbb{R}^2 - n \text{ pts}) \cong \pi_1(\mathbb{R}^3 - \text{unlink}) \cong F_n.$

$\pi_1(\text{graph}) \cong F_n.$

② $\pi_1(S^n) = 1 \quad n \geq 2.$

③ $\pi_1(S^3 - (p,q)\text{-torus knot}) \cong \langle x, y \mid x^p = y^q \rangle$

gluing two solid tori along an annulus.



VAN KAMPEN VIA PRESENTATIONS.

$$G_1 \cong \langle S_1 \mid R_1 \rangle$$
$$G_2 \cong \langle S_2 \mid R_2 \rangle$$
$$\Rightarrow G_1 * G_2 \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

What is a presentation for $\pi_1(A) * \pi_1(B) / N$?

First, a given $\gamma \in \pi_1(A \cap B)$ gives two elements of $\pi_1(A) * \pi_1(B)$:

$$\begin{tikzcd} & \pi_1(A) \arrow[dr] & \\ \pi_1(A \cap B) \arrow[ur] \arrow[dr] & & \pi_1(A) * \pi_1(B) \\ & \pi_1(B) \arrow[ur] & \end{tikzcd}$$

call them γ_A & γ_B.

Choose a generating set S for $\pi_1(A \cap B)$.
Choose presentations:

$$\pi_1(A) \cong \langle S_1 \mid R_1 \rangle$$
$$\pi_1(B) \cong \langle S_2 \mid R_2 \rangle$$

so each S_i contains each γ_A or γ_B for $\gamma \in S$.

Then:

$$\pi_1(A) * \pi_1(B) / N \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R \rangle$$

where R is the set of relations

$$\gamma_A = \gamma_B$$

for $\gamma \in S$.

<u>Proof</u> ① Let $f: I \to X$ loop at x_0.

Choose $0 = s_0 < s_1 < \cdots < s_m = 1$
s.t. $f|_{[s_i, s_{i+1}]}$ is a path in either A or B;
call it f_i.

$\forall i$, choose path g_i in $A \cap B$ from x_0 to $f(s_i)$

The loop

$$(f_1 \bar{g}_1)(g_1 f_2 \bar{g}_2) \cdots (g_{m-1} f_m)$$

is homotopic to f, and is a composition of loops, each in A or B. $\Rightarrow f \in \operatorname{Im} \Phi$.

② A <u>factorization</u> of $f \in \pi_1(X)$ is an element of $\Phi^{-1}(f)$:

$$f_1 \cdots f_m \qquad f_i \in \pi_1(A) \text{ or } \pi_1(B)$$

We showed in ① that each f has a factorization.

Two factorizations are equivalent modulo N iff they differ by a sequence of moves:

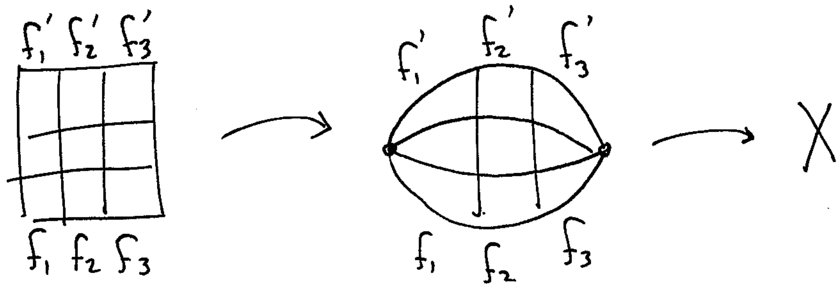
(i) Combine $[f_i][f_{i+1}] \rightsquigarrow [f_i f_{i+1}]$
if f_i, f_{i+1} lie both in $\pi_1(A)$ or in $\pi_1(B)$.

(ii) Regard $[f_i] \in \pi_1(A)$ as $[f_i] \in \pi_1(B)$
if $f_i \in \pi_1(A \cap B)$.

Let $f_1 \cdots f_k$, $f'_1 \cdots f'_\ell$ factorizations of f.
To show they are related by (i) & (ii).

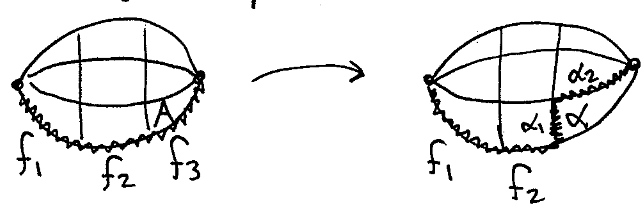
Choose a homotopy $I \times I \to X$ from one to the other.

Cut $I \times I$ into small rectangles, each mapping to A or B, and so induced partitions of top & bottom edges are finer than those coming from the factorizations.



Push across one square at a time. Show the new factorization differs from old by (i) & (ii).

E.g. two bottom-right squares.



$\alpha = f_3$ in $\pi_1(A)$

Then rewrite α as $\alpha_1 \alpha_2$ (move (i)).

rewrite α_1 as $\beta_1 \in \pi_1(B)$ (move (ii)).

Homotope $f_2 \beta_1 \in \pi_1(B)$ across square. etc. ▨

Attaching Disks

X path connected, based at x_0.
Attach 2-cell D^2 via $\varphi : S^1 \to X$.
$\rightsquigarrow Y$.
Choose path γ from x_0 to $\varphi(S^1)$.
The loop $\gamma\,\varphi(S^1)\,\bar{\gamma}$ is nullhomotopic in Y.
Let N = normal subgroup of $\pi_1(X)$ generated by this loop. Note: N independent of γ.

<u>Prop</u>. The inclusion $X \to Y$ induces a surjection
$$\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$$
with kernel N.

<u>Proof</u>: Choose $y \in \mathrm{int}(D^2)$
Apply Van Kampen to $Y - y$, $Y - X$.
Note: $Y - y \simeq X$
$Y - X \simeq *$
$(Y-y) \cap (Y-X) = \mathrm{int}(D^2) - y \simeq S^1$. ▨

<u>Applications</u>. ① M_g = orientable surface of genus g.
$$\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1,b_1]\cdots[a_g,b_g] = 1 \rangle$$
$\Rightarrow M_g \not\simeq M_h$ $g \neq h$ as
$\pi_1(M_g)^{ab} \cong \mathbb{Z}^{2g}$.

(2) For any group G, there is a 2-dim cell complex X_G with $\pi_1(X_G) \cong G$.

To do this, choose a presentation

$$G = \langle g_\alpha \mid r_\beta \rangle$$

$X_G = \bigvee_\alpha S^1$ with 2-cells attached along r_β.