CLASSIFICATION OF COVERING SPACES

$$\{\text{based covers of } X\} \leftrightarrow \{\text{subgroups of } \pi_1(X)\}$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

First step: find a cover corresponding to trivial subgroup.

Theorem: X = CW-complex (or any path conn, locally path conn, semilocally simply conn.) Then X has a universal cover $\tilde{X}$.

Proof: We construct $\tilde{X}$ directly.

Points in $\tilde{X}$ $\leftrightarrow$ homotopy classes of paths from $\tilde{x}_0$ (simple connectivity)
$\leftrightarrow$ homotopy classes of paths from x_0 (homotopy lifting)

So define:

$$\tilde{X} = \{ [\gamma] : \gamma \text{ a path in } X \text{ at } x_0 \}$$

$$p : \tilde{X} \longrightarrow X$$
$$[\gamma] \longmapsto \gamma(1)$$

Topology on $\tilde{X}$

$\mathcal{U} = \{ U \subseteq X : U \text{ path conn.}, \pi_1(U) \to \pi_1(X) \text{ trivial} \}$

For $U \in \mathcal{U}$, γ with $\gamma(1) \in U$, define

$$U_{[\gamma]} = \{ [\gamma \cdot \eta] : \eta \text{ a path in } U, \ \eta(0) = \gamma(1) \}$$
$$= \text{open neighborhood of } [\gamma] \text{ in } \tilde{X}.$$

exercise: The $U_{[\gamma]}$ form a basis.

We now check the properties of a covering space.

- Continuity. $p^{-1}(U)$ is a union of $U_{[\gamma]}$

- Path connectivity. Let $[\gamma] \in \tilde{X}$.

$$\gamma_t = \begin{cases} \gamma & \text{on } [0,t] \\ \text{const.} & \text{on } [t,1] \end{cases}$$

is a path from $[\text{const}]$ to $[\gamma]$.

- Simple connectivity. p_* injective, so suffices to show

$$p_* \pi_1(\tilde{X}) = 1.$$

Let $\gamma \in \operatorname{Im} p_* \Rightarrow \gamma$ lifts to a loop.

The lift of γ is $\{ [\gamma_t] \}$

loop $\Rightarrow [\gamma_1] = [\gamma_0]$

or $[\gamma] = [\text{const}]$

$\Rightarrow \gamma = 1$ in $\pi_1(X)$.

· Covering Space. Note: If $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$
Thus, for fixed $U \in \mathcal{U}$, the $U_{[\gamma]}$ partition $p^{-1}(U)$

$p: U_{[\gamma]} \longrightarrow U$ homeomorphism since it gives a bijection of open sets
$$V_{[\gamma]} \subseteq U_{[\gamma]} \longleftrightarrow V \subseteq U$$
for $V \in \mathcal{U}$. ▨

<u>Theorem</u>: For every $H \leq \pi_1(X)$ there is a (based) covering space
$$p: \tilde{X}_H \longrightarrow X$$
with $p_* \pi_1(\tilde{X}_H, \tilde{x}_0) = H$.

<u>Proof</u>: We realize $\tilde{X}_H$ as a quotient $\tilde{X}_H = \tilde{X}/\sim$:
$$[\gamma] \sim [\gamma'] \quad \text{if} \quad \gamma(1) = \gamma'(1) \quad \text{and} \quad [\gamma \cdot \bar{\gamma}'] \in H.$$
exercise: $\sim$ is an equivalence relation.

Check $\tilde{X}_H$ a covering space:
Say $[\gamma] \sim [\gamma']$ with $\gamma(1) = \gamma'(1) \in U \in \mathcal{U}$.
Then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$ for any path η in U.
$\Rightarrow$ $U_{[\gamma]}$ identified with $U_{[\gamma']}$

Check $p_* \pi_1(\tilde{X}_H) = H$:
Let $\tilde{x}_0 = [\text{const}]$.
$\gamma \in \operatorname{Im} p_* \iff \{[\gamma_t]\}$ a loop in $\tilde{X}_H$
$\iff [\gamma_0] \sim [\gamma_1]$
i.e. $[\text{const}] \sim [\gamma]$
$\iff \gamma \in H$. ▨

To finish classification, need to show $\tilde{X}_H$ unique.

Def: Covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ with $p_1 = p_2 f$ (i.e. f preserves fibers).

Prop: Two path connected covering spaces $p_1 : (\tilde{X}_1, \tilde{x}_1) \to X$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \to X$ are isomorphic if and only if

$$\operatorname{Im}(p_1)_* = \operatorname{Im}(p_2)_*.$$

Proof: $\Rightarrow$ easy.

$\Leftarrow$ Lifting criterion $\rightsquigarrow$ lift p_1 to $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \tilde{p}_1 = p_1$

By symmetry $\rightsquigarrow$ $\tilde{p}_2$ with $p_1 \tilde{p}_2 = p_2$.

Note $\tilde{p}_1 \tilde{p}_2$ is a lift of p_2:

$$p_2 \tilde{p}_1 \tilde{p}_2 = p_1 \tilde{p}_2 = p_2$$

Unique lifting + $\tilde{p}_1 \tilde{p}_2 (\tilde{x}_2) = \tilde{x}_2 \Rightarrow \tilde{p}_1 \tilde{p}_2 = id$.

Symmetry: $\tilde{p}_2 \tilde{p}_1 = id$.

$\Rightarrow \tilde{p}_1$ a homeo. ▨

Cor: Every subgroup of a free group is free.

Some Examples of Covering Spaces

$$S^1 \times \mathbb{R} \longrightarrow T^2$$

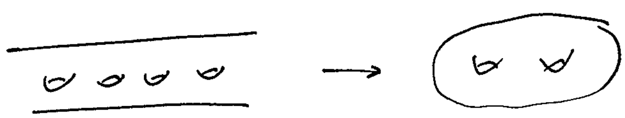
$$T^2 \xrightarrow{(\times m, \times n)} T^2$$

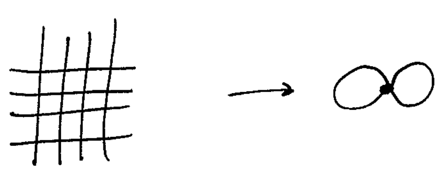
Annulus $\longrightarrow$ Möbius strip

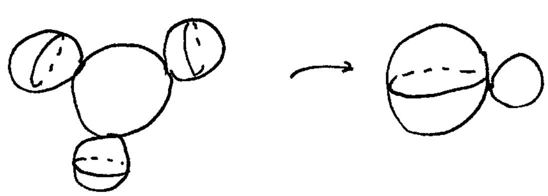
$$S^2 \longrightarrow \mathbb{RP}^2$$

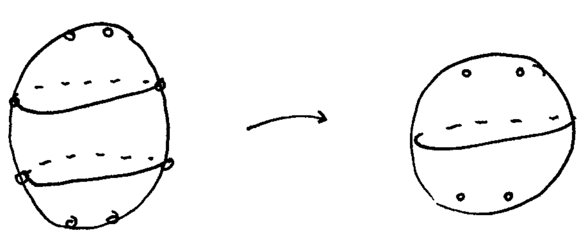
$$\mathbb{C}^* \xrightarrow{z^n} \mathbb{C}^*$$

$$\mathbb{C}^* \longrightarrow T^2$$









THE FUNDAMENTAL THEOREM

Fix $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$
$H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$
$N(H) =$ normalizer in $\pi_1(X, x_0)$
$G(\tilde{X}) =$ group of deck transformations.

Say p is <u>regular</u> if $G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$.

Regard $\tilde{x}_0$ as [const]
Then $p^{-1}(x_0) = \{ [\gamma] : \gamma \text{ a loop} \}$
By lifting criterion,

$\exists$ deck trans τ_γ taking [const] to $[\gamma]$
$\Leftrightarrow p_* \pi_1(\tilde{X}, [\gamma]) = p_* \pi_1(\tilde{X}, [\text{const}])$
or $\gamma \, p_* \pi_1(\tilde{X}, [\text{const}]) \, \gamma^{-1} = p_* \pi_1(\tilde{X}, [\text{const}])$
i.e. $\gamma \in N(H)$.

We thus have:

$$\begin{aligned} N(H) &\longrightarrow G(\tilde{X}) \\ \gamma &\longmapsto \tau_\gamma \end{aligned}$$

Note: well-defined by uniqueness of lifts.

<u>Prop</u>: $\tilde{X}$ regular $\Leftrightarrow$ H normal.

<u>Theorem</u>: $G(\tilde{X}) \cong N(H)/H$

Both are exercises.

Covering Spaces via Actions

An action of a group G on a space Y is a homom:

$$G \longrightarrow \text{Homeo}(Y)$$

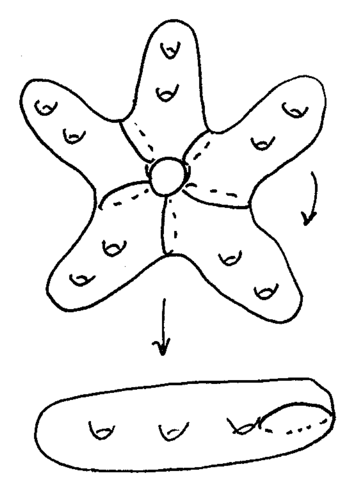
This is a covering space action if
$\forall\, y \in Y\ \exists$ neighborhood U with
$\{g(U) : g \in G\}$
all distinct, disjoint.

Fact: The action of $G(\tilde{X})$ on $\tilde{X}$ is a covering space action.

Prop: Y = connected CW-complex
(or any path conn, locally path conn)
$G \circlearrowright Y$ via covering space action. Then:
(i) $p : Y \to Y/G$ a regular covering space.
(ii) $G \cong G(Y)$

In particular
- $G \cong \pi_1(Y/G) / p_* \pi_1(Y)$
- Y simply connected $\Rightarrow \pi_1(Y/G) \cong G$.

Examples.
$\mathbb{Z} \circlearrowright \mathbb{R} \rightsquigarrow S^1$
$\mathbb{Z} \circlearrowright \mathbb{R} \times I \rightsquigarrow$ Möbius strip
$\mathbb{Z}^2 \circlearrowright \mathbb{R}^2 \rightsquigarrow T^2$
Klein bottle
$\mathbb{Z}/2\mathbb{Z} \circlearrowright S^n \rightsquigarrow \mathbb{RP}^n$
$\mathbb{Z}/m\mathbb{Z} \circlearrowright M_{mk+1} \longrightarrow M_{k+1}$



$K(G,1)$ Spaces

Goal: groups $\longleftrightarrow$ spaces (up to homotopy equiv.)
homomorphisms $\longleftrightarrow$ continuous maps (up to homotopy)

A $K(G,1)$ space is a space with fundamental group G and contractible universal cover.

Examples. S^1, T^2 in general $\mathbb{Z}^n \leftrightarrow T^n$

What about $G = \mathbb{Z}/m\mathbb{Z}$?

$\mathbb{Z}/m\mathbb{Z}$ acts on S^∞ = unit sphere in $\mathbb{C}^\infty$ via
$$(z_i) \mapsto e^{2\pi i m}(z_i)$$
which is a covering space action.
(When $m=2$, quotient is $\mathbb{RP}^\infty$).

Why is S^∞ contractible?
Step 1: $f_t(x_1, x_2, \ldots) = (1-t)(x_i) + t(0, x_1, x_2, \ldots)$
Step 2: Straight line projection to $(1,0,0,\ldots)$.

Later: Any $K(\mathbb{Z}/m\mathbb{Z},1)$ is ∞-dim!

CONSTRUCTION OF $K(G,1)$ spaces

<u>Prop</u>: Every group G has a $K(G,1)$

<u>Proof</u>: Define a Δ-complex EG with:

$$n\text{-simplices} \longleftrightarrow \begin{array}{l}\text{ordered}\\ (n+1)\text{-tuples}\\ [g_0, \ldots, g_n] \quad g_i \in G\end{array}$$

To see EG contractible, slide each $x \in [g_0, \ldots, g_n]$ along line segment in $[e, g_0, \ldots, g_n]$ from x to $[e]$

(Note: This is not a deformation retraction since it moves $[e]$ around $[e,e]$.)

$G \circlearrowright EG$ by left multiplication.
exercise: This is a covering space action.

$\rightsquigarrow BG = EG/G$ is a $K(G,1)$.

This gives one $K(G,1)$, and it is always ∞-dim.
To study a group G, need a <u>good</u> $K(G,1)$,
e.g. $K(PB_n,1) = \mathbb{C}^n \setminus \Delta$.

Homomorphisms as Maps

<u>Prop</u>: X = connected CW-complex
$Y = K(G,1)$
Every homomorphism $\pi_1(X,x_0) \longrightarrow \underset{}{\overset{\pi_1(Y,y_0)}{\overset{\shortparallel}{G}}}$ is induced by a map $(X,x_0) \longrightarrow (Y,y_0)$.
The map is unique up to homotopy fixing y_0.

This implies:

<u>Prop</u>: The homotopy type of a CW-complex $K(G,1)$ is uniquely determined by G.

<u>Proof of</u> <u>1st Prop</u>: Assume first X has one 0-cell, x_0.

Let $\varphi: \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$. Want $f: X \to Y$.

Step 0. $f(x_0) = y_0$

Step 1. Each edge e of X is an element of $\pi_1(X,x_0)$. Define $f(e)$ via φ.

Step 2. Let Δ = 2-cell with $\psi: \partial\Delta \longrightarrow X^{(1)}$
$f\psi$ null-homotopic, since φ a homom.
$\rightsquigarrow$ can extend f to Δ.