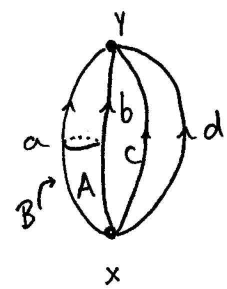
Homology

Fundamental groups are good at telling spaces apart, but it is not so easy to compute, and the higher dimensional analogs are very hard to compute. Indeed: computing $\pi_m(S^n)$ is a huge open problem.

Homology is an analogue that is computable. We will lose some information, but it will still be possible to tell many spaces apart.

Example. $X =$ 

A = front of sphere
B = back.

C_0 = free abelian group on x, y
C_1 = free abelian group on a, b, c, d
C_2 = free abelian group on A, B.

An element of $H_1(X)$ is a 1-cycle: an element of C_1 with no boundary, e.g. ab^{-1}.

Since C_1 abelian, $ab^{-1} = b^{-1}a$ so we think of ab^{-1} as a loop with no basepoint.

A 1-cycle is trivial if it is the boundary of a 2-cell, or a collection of 2-cells, so:

ab^{-1} trivial, cd^{-1} not.

In other words, $H_1(X) = {}^{\text{1-cycles}}/_{\text{1-boundaries}}$.

Can compute with linear algebra.

$$\partial_1 : C_1 \to C_0 \qquad \text{“boundary map”}$$
$$a,b,c,d \mapsto y-x$$

$$\text{1-cycles} = \ker \partial_1.$$

$$\partial_2 : C_2 \to C_1$$
$$A,B \mapsto a-b$$

$$\text{1-boundaries} = \operatorname{im} \partial_2.$$

So: $H_1(X) = {}^{\ker \partial_1}/_{\operatorname{im} \partial_2}$

Exercise: $\ker \partial_1 = \langle a-b,\ b-c,\ c-d \rangle \cong \mathbb{Z}^3$
$\operatorname{im} \partial_2 = \langle a-b \rangle$

essentially lin. alg.

$\Rightarrow H_1(X) \cong \mathbb{Z}^2$

Also: $H_2(X) = {}^{\ker \partial_2}/_{\operatorname{im} \partial_3} = {}^{\langle A-B \rangle}/_{1} \cong \mathbb{Z}$.

Simplicial Homology

$X = \Delta$-complex

$\Delta_n(X) =$ free abelian group on n-simplices of X.

$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

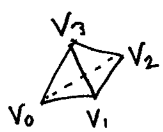
$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

There is also singular homology: $X =$ any space

$C_n(X) =$ free abelian group on all maps $\Delta^n \to X$.

More complicated, but more powerful. Will turn out to be equivalent.

Δ-complexes

Simplex 

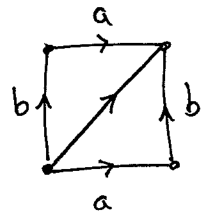
ordering of vertices $\rightsquigarrow$ ordering of vertices for each face.

To build a Δ-complex:

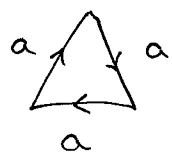
- Start with a discrete set of vertices
- Attach edges to produce a graph.
- Attach 2-simplices along edges, respecting orderings of vertices
- etc.

$\Delta_n(X) =$ free abelian group on n-simplices.

Exercise: every simplicial complex has the structure of a Δ-complex.

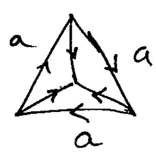
Examples. ①  $\cong T^2$

②  Dunce hat.

③ 

Nonexample. No way to order vertices of 2-simplex consistently with edges.

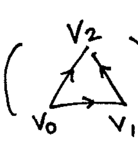
Here is a Δ-complex structure on same space:



Boundary homomorphism

$$\partial([v_0,\dots,v_n]) = \sum (-1)^i [v_0,\dots,\hat{v}_i,\dots,v_n]$$

e.g. $\partial([v_0,v_1,v_2]) = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$

or: $\partial\Big($  $\Big) =$ 

where $[v_0,\dots,v_n]$ is Δ^n = standard n simplex.

For a simplex $\sigma : \Delta^n \to X$ in Δ-complex:

$$\partial\sigma(\Delta^n) = \sigma(\partial\Delta^n).$$

Lemma: $\partial_{n-1} \circ \partial_n = 0$.

Proof: Check on one simplex $\Delta = [v_0, \dots, v_n]$

$$\partial_n(\Delta) = \sum (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

$$\partial_{n-1}\partial_n(\Delta) = \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

$$+ \sum_{j>i} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= 0. \qquad \text{(switch roles of } i \text{ \& } j \text{ in last sum).}$$

We now have:

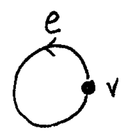
$$\cdots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \longrightarrow 0$$

with $\partial_n \partial_{n+1} = 0 \ \forall n$. i.e. $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$

This is called a chain complex.

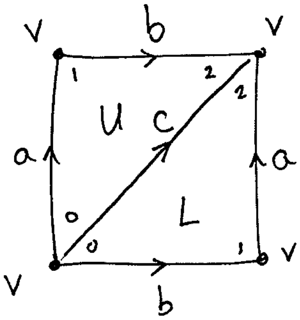
$\rightsquigarrow$ can define: $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1} = $ n-cycles / n-boundaries

"n^{th} homology group of X"

EXAMPLES. ① $X = S^1 =$ 

$$\Delta_0(X) = \langle v \rangle \cong \mathbb{Z}$$
$$\Delta_1(X) = \langle e \rangle \cong \mathbb{Z}$$
$$\partial_1 = 0 \qquad \partial_1(e) = v - v = 0.$$
$$\rightsquigarrow \quad H_n(X) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

② $X = T^2$

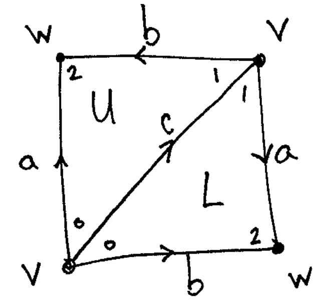


$$\partial_1 = 0 \qquad \partial_0 = \partial_3 = 0.$$
$$\partial_2(U) = \partial_2(L) = a + b - c$$

$$H_0(X) = \langle v \rangle / 0 \cong \mathbb{Z}$$

$$H_1(X) = \langle a,b,c \rangle / \langle a+b-c \rangle \cong \langle a,b \rangle \cong \mathbb{Z}^2$$
$$H_2(X) = \langle U - L \rangle / 0 \cong \mathbb{Z}.$$

③ $X = \mathbb{RP}^2$



$$H_0(X) = \langle v, w \rangle / \langle v - w \rangle = \mathbb{Z}$$

$$\ker \partial_1 = \langle a-b, c \rangle = \langle c, a-b+c \rangle \cong \mathbb{Z}^2$$

$$\operatorname{im} \partial_2 = \langle a+b+c, a-b+c \rangle = \langle a-b+c, 2c \rangle \cong \mathbb{Z}^2$$

$$\rightsquigarrow H_1(X) = \langle c, a-b+c \rangle / \langle 2c, a-b+c \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

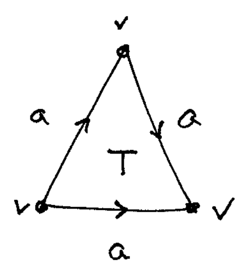
Next: $\ker \partial_2$

$$\partial_2(pU + qL) = (q-p)a + (p-q)b + (p+q)c$$

$$\Longrightarrow \ker \partial_2 = 0.$$

$$\rightsquigarrow H_2(X) = 0.$$

④ X = Dunce cap



X is contractible but not collapsible.

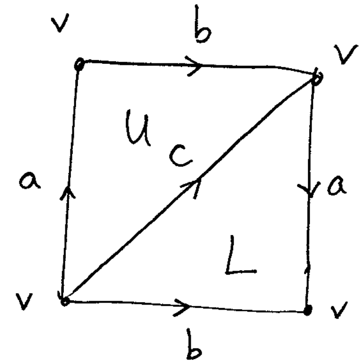
$$H_1(X) = \langle a \rangle / \langle a \rangle = 0$$

$$H_0(X) = \langle v \rangle / 0 \cong \mathbb{Z}$$

$$H_2(X) = 0$$

Exercise: $X \simeq *$ (it is mapping cone of deg 1 map $S^1 \to S^1$).

⑤ X = Klein bottle



$H_0(X) = \langle v \rangle / 0 \cong \mathbb{Z}$
$H_2(X) = 0.$

$\operatorname{Ker} \partial_1 = \langle a, b, c \rangle$
$\operatorname{Im} \partial_2 = \langle a+b-c, a-b+c \rangle$

How to compute quotient? Find Smith normal form of:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

i.e. use row/col ops to get diagonal matrix where each diagonal entry divides the next.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$H_1(X) \cong \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/0\mathbb{Z}$
$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$

Will prove: $H_1(X) \cong \pi_1(X)^{ab}$.