Singular Homology

Simplicial homology is very computable, but:

① It is not obvious that homeomorphic Δ-complexes have isomorphic simplicial homology.

② Hard to prove general facts about spaces.

So: A <u>singular</u> n-<u>simplex</u> in X is a map $\sigma : \Delta^n \to X$

Let $C_n(X)$ = free abelian group on these.

$= $ group of n-chains

$= \{ \sum n_i \sigma_i \mid n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \}$

Boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\sigma \mapsto \sum (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$$

Still have $\partial_{n-1} \circ \partial_n = 0$.

$$H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$ "nth singular homology group"

Singular homology hard to compute. For example, not obvious that

① $H_n(X) = 0$ for $n > \dim X$

② $H_n(X)$ finitely gen.

On other hand, easy to prove general facts like:

<u>Fact</u>: Homeomorphic spaces have isomorphic singular hom. groups.

Will show: singular = simplicial.

<u>Note</u>. Elements of $H_1(X)$ rep. by maps $S^1 \to X$ (easy)

$H_2(X)$ rep. by maps $M_g \to X$ (less easy)

$H_n(X)$ rep by maps n-manifold $\to X$ (only true over $\mathbb{Q}$)

Prop: X = space with path components X_α
$\Rightarrow H_n(X) \cong \bigoplus H_n(X_\alpha)$

Prop: X = nonempty, path conn. $\Rightarrow H_0(X) \cong \mathbb{Z}$
X has n path comp. $\Rightarrow H_0(X) \cong \mathbb{Z}^n$

Proof: Say X path conn.

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$H_0(X) = C_0(X)/\operatorname{Im}\partial_1$

Given $v, w \in X$, $v - w \in \operatorname{Im}\partial_1 \Rightarrow v = w$ in $H_0(X)$.

Also, $nv \neq 0$ in $H_0(X)$ since $\operatorname{Im}\partial_1 \subseteq \ker(C_0(X) \xrightarrow{\varepsilon} \mathbb{Z})$

where $\varepsilon(\sum n_i v_i) = \sum n_i$. ∎

Prop: X = pt.

$$\Rightarrow H_i(X) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

Pf: $C_n(X) \cong \mathbb{Z}\ \forall n$.

$$\partial(\sigma_n) = \sum(-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$ ∎

Reduced Homology

Looking at last Prop, seems more elegant to replace last 0 map with $\cong$.

$\tilde{H}_n(X)$ = homology of $\cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

where $\varepsilon(\sum n_i \sigma_i) = \sum n_i$

= reduced homology of X.

Exercise: $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$

In particular: $\tilde{H}_i(X) = 0\ \forall i$ when X = pt.

Homotopy Invariance

Goal: $f : X \to Y \rightsquigarrow f_* : H_n(X) \longrightarrow H_n(Y)$

and

f homotopy equivalence $\Rightarrow$ f_* an isomorphism.

First, $f \rightsquigarrow f_\# : C_n(X) \longrightarrow C_n(Y)$

$$\sigma \longmapsto f\sigma$$

with $f_\# \partial = \partial f_\# \rightsquigarrow$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

"chain map"

$f_\#$ takes cycles to cycles, boundaries to boundaries.

$\Rightarrow$ $f_\#$ induces $f_* : H_n(X) \longrightarrow H_n(X)$

Facts: $(fg)_* = f_* g_*$

$\quad\quad\; id_* = id$

Theorem. $f, g : X \to Y$ homotopic $\Rightarrow f_* = g_*$

Cor: $f : X \to Y$ homotopy equiv. $\Rightarrow f_*$ an isomorphism.

example. X contractible $\Rightarrow \tilde{H}_i(X) = 0 \;\; \forall\, i$.

Proof of Theorem: We will define $P : C_n(X) \to C_{n+1}(Y)$ with

$$\partial P = g_\# - f_\# - P\partial \qquad \text{"prism operator"}$$

P is the homotopy from $f\sigma$ to $g\sigma$.

The theorem follows:

If $\alpha \in C_n(Y)$ is a cycle, then

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\overbrace{\partial(\alpha)}^{0} = \partial P(\alpha)$$

$\Rightarrow (g_\# - f_\#)(\alpha)$ a boundary

$\Rightarrow \cancel{g_\#(\alpha)}\ g_*(\alpha) = f_*(\alpha)$

Remains to define P and check $\partial P = g_\# - f_\# - P\partial$.

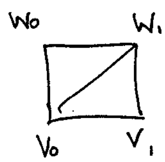
Main ingredient. Cutting $\Delta^n \times I$ into $(n+1)$-simplices

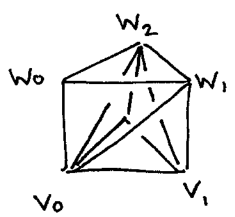
Label vertices of $\Delta^n \times 0$ by $v_0, \dots, v_n$

$\Delta^n \times 1$ by $w_0, \dots, w_n$.

$\Delta^n \times I$ decomposes as sum of

$[v_0, \dots, v_i, w_i, \dots, w_n]$





Define $P(\sigma) = \sum (-1)^i F \circ (\sigma \times id)\big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$

where F = homotopy from f to g.

and $\Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$

exercise: $\partial P = g_\# - f_\# - P\partial$ (like proof that $\partial_n \circ \partial_{n+1} = 0$).

▨

The relationship $\partial P + P\partial = g_\# - f_\#$ is expressed as:

P is a chain homotopy from $f_\#$ to $g_\#$

__Prop__: Chain homotopic maps between exact sequences induce the same map on homology.

Exact Sequences

A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

is <u>exact</u> if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$

<u>chain complex</u> if $\operatorname{im} \alpha_{n+1} \subseteq \ker \alpha_n$ i.e. $\alpha_n \circ \alpha_{n+1} = 0$.

Facts:
(i) $0 \longrightarrow A \xrightarrow{\alpha} B \iff \alpha$ injective
(ii) $A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ surjective
(iii) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ isomorphism.
(iv) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \iff C \cong B/A$ "short exact sequence"

Collapsing a Subcomplex

Theorem. $(X, A) =$ CW-pair.

There is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}(X/A)$$

$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{q_*} \tilde{H}_{n-1}(X/A) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0.$$

where $i : A \hookrightarrow X$, $q : X \to X/A$.

Cor: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

Proof: Induction on n.

$\tilde{H}_0(S^0) \cong \mathbb{Z}$ ✓

For $n > 0$: $(X, A) = (D^n, S^{n-1}) \rightsquigarrow X/A \cong S^n$.

By theorem:

$$\cdots \longrightarrow \tilde{H}_i(D^n) \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \tilde{H}_{i-1}(D^n) \longrightarrow \cdots$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$ ∎

To prove the Theorem, will do something more general...

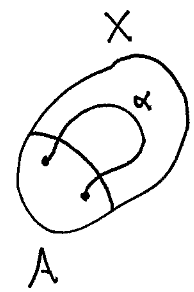
Relative Homology

$A \subseteq X \rightsquigarrow C_n(X,A) \cong C_n(X)/C_n(A)$

Since ∂ takes $C_n(A)$ to $C_{n-1}(A)$, have chain complex

$$\cdots \to C_n(X,A) \to C_{n-1}(X,A) \to \cdots$$

$\rightsquigarrow$ relative homology groups $H_n(X,A)$.



Elements of $H_n(X,A)$ are rep by relative cycles:

$$\alpha \in C_n(X) \quad \text{s.t. } \partial\alpha \in C_{n-1}(A)$$

A relative cycle is trivial in $H_n(X,A)$ iff it is a relative boundary:

$$\alpha \in C_n(X) \quad \alpha = \partial\beta + \gamma \quad \text{some } \beta \in C_{n+1}(X), \gamma \in C_n(A)$$

Will show: $H_n(X,A) \cong H_n(X/A)$.

Goal: Long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A)$$

$$\to H_{n-1}(A)$$

Proof is "diagram chasing".

To start :

$$
\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_*"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n(X) \arrow[r, "q_*"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_*"] & C_{n-1}(X) \arrow[r, "q_*"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$\rightsquigarrow$ Short exact sequence of chain complexes:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d, "i"] & C_n(A) \arrow[r] \arrow[d] & C_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "q"] & C_n(X) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X,A) \arrow[r] \arrow[d] \arrow[uur, dashed, bend right] & C_n(X,A) \arrow[r] \arrow[d] \arrow[uur, dashed, bend right] & C_{n-1}(X,A) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Commutativity of squares $\Rightarrow$ i_*, q_* chain maps.

$\rightsquigarrow$ induced maps on homology.

Need to define $\partial : H_n(X,A) \to H_{n-1}(A)$

Let $c \in C_n(X,A)$ a cycle.

$c = q(\tilde{c})$ $\tilde{c} \in C_n(X)$

$\partial\tilde{c} \in \ker q$ by commutativity.

$\Rightarrow$ $\tilde{c} = i(a)$ some $a \in C_{n-1}(A)$ by exactness.

and $\partial a = 0$ by commut: $i\partial(a) = \partial i(a) = \partial\partial(\tilde{c}) = 0$.

i inj.

Set $\partial[c] = [a] \in H_{n-1}(A)$.

Claim: $\partial : H_n(X,A) \to H_{n-1}(A)$ is a well-defined homomorphism.

Well-defined. · a determined by $\partial\tilde{c}$ since i injective

· different choice $\tilde{c}'$ for $\tilde{c}$ would have

$\tilde{c}' - \tilde{c} \in C_n(A)$, i.e. $\tilde{c}' = \tilde{c} + i(a')$

$\Rightarrow$ a changes to $a + \partial a'$

since $i(a+\partial a') = i(a) + i(\partial a') = \partial\tilde{c} + \partial i(a') = \partial(\tilde{c} + i(a'))$

· different choice for $\tilde{c}$ in $[c]$ is of form $c + \partial c'$

Since $c' = q(b')$ some b' $\rightsquigarrow$ $c + \partial c' = c + \partial j(b')$

$b' \rightsquigarrow \tilde{c}'$

$= c + q(\partial b') = q(\tilde{c} + \partial b')$

so $\tilde{c}$ replaced by $\tilde{c} + \partial b'$

$\rightsquigarrow$ $\partial\tilde{c}$ unchanged.

Homomorphism. Say $\partial[c_1] = [a_1]$, $\partial[c_2] = [a_2]$ via $\tilde{c}_1, \tilde{c}_2$.

Then $q(\tilde{c}_1 + \tilde{c}_2) = c_1 + c_2$

$i(a_1 + a_2) = \partial(\tilde{c}_1 + \tilde{c}_2)$

so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$ //

Theorem. The following sequence is exact:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Proof. More diagram chasing. We'll do 2 of the 6 inclusions needed.

$\operatorname{Im}\partial \subset \ker i_*$ i.e. $i_*\partial = 0$:

$i_*\partial$ takes $[c]$ to $[\partial\tilde{c}] = 0$.

$\ker i_* \subset \operatorname{Im}\partial$: Say $a \in C_{n-1}(A)$, $a \in \ker i_* \Rightarrow i(a) = \partial b$ $b \in C_n(X)$

$\Rightarrow q(b)$ a cycle since $\partial q b = q \partial b = q i(a) = 0$.

& ∂ takes $[q(b)]$ to $[a]$. ▨