Product Structures

There are three natural products with homology & cohomology:

① Evaluation pairing:

$$H^k(X) \times H_k(X) \longrightarrow \mathbb{Z}$$

Can use this to show cocycles, or cycles, are nontrivial!

② Cup product:

$$H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$
$$(\varphi, \psi) \longmapsto \varphi \cup \psi$$

$\rightsquigarrow H^*(X)$ is a graded ring.

③ Cap product:

$$H^p(X) \times H_n(X) \longrightarrow H_{n-p}(X)$$
$$(\varphi, \alpha) \longmapsto \varphi \cap \alpha$$

Big Goal:

Poincaré Duality Theorem. Let M = compact, connected, oriented n-manifold. Then

$$H^p(M) \longrightarrow H_{n-p}(M)$$
$$\varphi \longmapsto \varphi \cap [M]$$

is an isomorphism.

We have already since examples of p-cocycles in n-manifolds of the form "intersect with this $(n-p)$-cycle". These are Poincaré duals.

Will see: under PD, cap product is intersection.

Cup Product

Want to define a product on $H_*(X)$.

There is a cross product $H_i(X) \times H_j(Y) \longrightarrow H_{i+j}(X \times Y)$

$$(e_i, e_j) \longmapsto e_i \times e_j$$

Taking $X = Y$: $H_i(X) \times H_j(X) \longrightarrow H_{i+j}(X \times X) \xrightarrow{?} H_{i+j}(X)$

Need a natural map $X \times X \to X$.

If X is a group, can multiply $\rightsquigarrow$ Pontryagin product.

Otherwise only natural map is projection $\rightsquigarrow$ stupid product.

For H^*, situation is better. Want

$$\cancel{H^i(X) \times H^j(X) \to H^{i+j}(X) \xrightarrow{?}}$$

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{?} H^{i+j}(X)$$

This requires a natural map $X \to X \times X$ $\rightsquigarrow$ diagonal!

This is the cup product.

We can also define cup product from scratch:

For $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$ $\qquad R = $ ring.

the cup product $\varphi \cup \psi \in C^{k+\ell}(X,R)$ is

given by : $\quad (\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$

for a simplex $\sigma : \Delta^{k+\ell} \to X$.

To show cup product induces a product on cohomology.

<u>Lemma</u> $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

<u>Pf</u> Say $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$, $\sigma: \Delta^{k+\ell+1} \to X$.

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|[v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}])\psi(\sigma|[v_{k+1},\ldots,v_{k+\ell+1}])$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|[v_0,\ldots,v_k])\psi(\sigma|[v_k,\ldots,\hat{v}_i,\ldots,v_{k+\ell+1}])$$

Last term of first sum cancels first sum of second.
Rest is $\delta(\varphi\cup\psi)(\sigma) = (\varphi\cup\psi)(\partial\sigma)$. ▨

Since $\delta(\varphi\cup\psi) = \delta\varphi\cup\psi \pm \varphi\cup\delta\psi$

$\rightsquigarrow$ product of cocycles is a cocycle.

Also, the product of a cocycle and a coboundary is a coboundary:

$$\psi = \delta\theta,\ \delta\varphi = 0 \implies \delta(\varphi\cup\theta) = \delta\varphi\cup\theta \pm \varphi\cup\delta\theta = \pm\varphi\cup\psi.$$

We thus have an induced cup product

$$H^k(X,R) \times H^\ell(X,R) \xrightarrow{\cup} H^{k+\ell}(X,R)$$

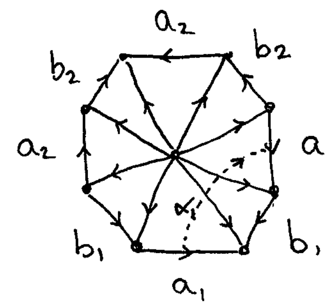
It is associative and distributive, since it is on cochain level.
If R has 1 then $H^*(X,R)$ has identity, namely:

$1 \in H^0(X,R)$ taking value $1 \in R$ on each 0-simplex.

Note: The canonical isomorphism between simplicial/singular H^* preserves $\cup$, so can switch back & forth.

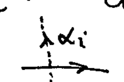
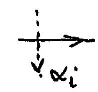
EXAMPLE: SURFACES

$X = M_g$. Will show $\cup : H^1(M_g, \mathbb{Z}) \times H^1(M_g, \mathbb{Z}) \longrightarrow H^2(M_g, \mathbb{Z}) = \mathbb{Z}$
is algebraic intersection.



a_i, b_i form a basis for $H_1(M_g, \mathbb{Z})$.
UCT $\Rightarrow$ $H^1(M_g) \cong \mathrm{Hom}(H_1(M_g), \mathbb{Z})$
Basis for H_1 $\rightsquigarrow$ dual basis for H^1

$$a_i \rightsquigarrow \varphi_i \quad \begin{array}{l} a_1 \mapsto 1 \\ \text{others} \mapsto 0. \end{array}$$

Can represent φ_i, ψ_i by simplicial cocycle $\rightsquigarrow$ dotted arc. α_i, β_i.
α_i evaluates to 1 on an edge like 
-1 on an edge like 

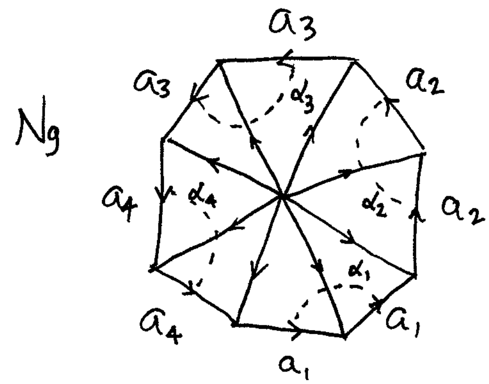
Compute $\varphi_1 \cup \psi_1$ from definition.
Takes value 0 on all cells but SE,
where it takes value 1.

We know $H_2(M_g) = \mathbb{Z} = \langle [M_g] \rangle$ ← fundamental class
UCT $\Rightarrow$ $H^2(M_g, \mathbb{Z}) \cong \mathrm{Hom}(H_2(M_g), \mathbb{Z})$.
So which elt of $H^2(M_g, \mathbb{Z})$ is $\varphi_1 \cup \psi_1$?
We check $(\varphi_1 \cup \psi_1)([M_g]) = 1$
This tells us both that (i) $[M_g]$ generates $H_2(M_g)$
(ii) $\varphi_1 \cup \psi_1$ is dual to $[M_g]$,
hence a gen. for $H^2(M_g, \mathbb{Z})$.

In general, identifying $H^2(M_g, \mathbb{Z})$ with $\mathbb{Z}$:
$$\cup = \hat{\imath}$$
↖ algebraic intersection.

Suffices to check on generators.

EXAMPLE: NONORIENTABLE SURFACES



Use $\mathbb{Z}/2\mathbb{Z}$ coefficients since

$H_2(N_g) = 0$

$H_2(N_g; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

Claim: $H^2(N_g, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Pf: Any single Δ gives a cocycle. φ

Any two adjacent triangles are cohomologous

$\rightsquigarrow$ any cocycle is $k\varphi$.

Can also use UCT and $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

Can check: $\alpha_i \cup \alpha_i = 1$

$\alpha_i \cup \alpha_j = 0$

This is again intersection number: if you push off α_i it intersects itself in one point.

The $g=1$ case is $\mathbb{R}P^2$.

$$\rightsquigarrow H^*(\mathbb{R}P^2) = \{1, \alpha, \alpha \cup \alpha\}$$
$$= \mathbb{Z}/2\mathbb{Z}[\alpha]/\langle \alpha^2 \rangle.$$

NATURALITY

Prop: For $f: X \to Y$, the induced $f^*: H^n(Y,R) \to H^n(X,R)$ satisfies:
$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

Pf: Already true on cochain level: $f^\#(\varphi) \cup f^\#(\psi) = f^\#(\varphi \cup \psi)$.
$$\begin{aligned}(f^\#(\varphi) \cup f^\#(\psi))(\sigma) &= f^\#\varphi(\sigma|_{[v_0,\dots,v_k]})\, f^\#\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\ &= \varphi(f\sigma|_{[v_0,\dots,v_k]})\, \psi(f\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\ &= (\varphi \cup \psi)(f\sigma) \\ &= f^\#(\varphi \cup \psi)(\sigma).\end{aligned}$$
∎

RELATIVE VERSION

$C^k(X,A;R)$ = cochains that vanish on A
(more natural than $C_k(X,A)$ since it is a subgroup, not a quotient).

Have cup products:
$$\left.\begin{array}{l} H^k(X;R) \times H^\ell(X,A;R) \\ H^k(X,A;R) \times H^\ell(X;R) \\ H^k(X,A;R) \times H^\ell(X,A;R) \end{array}\right\} \longrightarrow H^{k+\ell}(X,A;R)$$

And: $H^k(X,A;R) \times H^\ell(X,B;R) \longrightarrow H^{k+\ell}(X, A \cup B; R)$.

THE COHOMOLOGY RING

Define $H^*(X,R) = \bigoplus H^k(X,R)$
Elements are finite sums $\sum \alpha_i$ with $\alpha_i \in H^i(X,R)$.
The product is $\sum \alpha_i \sum \beta_j = \sum \alpha_i \beta_j$
(writing xy for $x \cup y$).
$\rightsquigarrow$ $H^*(X,R)$ is a ring. It has 1 if R has 1.

We saw: $H^*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) = \{a_0 + a_1\alpha + a_2\alpha^2 : a_i \in \mathbb{Z}/2\mathbb{Z}\}$
$= \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^3)$ nice!

One can also show: $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$. $\quad |\alpha| = 1$
$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]$
and $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ $\quad |\alpha| = 2$.
$H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[\alpha]$.

H^* is a graded ring, a ring of form $\bigoplus A_k$ with
A_k = additive subgroup, $A_k \times A_\ell \subseteq A_{k+\ell}$.
Write $|\alpha|$ for the degree (i.e. which A_k it lives in).

There are spaces with same H_k & H^k groups, but
different H^*: $S^1 \vee S^1 \vee S^2$, T^2

There are distinct spaces with *identical* H^*:
$H^*(S^3 \vee S^5) \cong H^*(S(\mathbb{CP}^2)) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(5)}$ degree.

Prop: $\alpha \cup \beta = (-1)^{k+\ell}(\beta \cup \alpha)$ if R commutative.

Künneth Formula

Next goal: $H^*(T^n, \mathbb{Z})$ = free abelian gp with basis

$$\alpha_{i_1} \cup \cdots \cup \alpha_{i_k} \qquad i_1 < \cdots < i_k$$

where $\alpha_i \in H^1(T^n, \mathbb{Z})$ is $p_i^*(\alpha)$ for α a gen of $H^1(S^1, \mathbb{Z})$.
and p_i is projection to ith factor.

Cross Product (aka external cup product)

$$H^*(X, \mathbb{Z}) \times H^*(Y, \mathbb{Z}) \longrightarrow H^*(X \times Y, \mathbb{Z})$$

$$(a, b) \longmapsto p_1^*(a) \cup p_2^*(b)$$

bilinear.

Tensor Products

Bilinear maps are not linear/homomorphisms

e.g. $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(e_1, e_1) \longmapsto 1$$

$$\Rightarrow (-e_1, -e_1) \longmapsto 1$$

$\rightsquigarrow$ replace $\times$ with $\otimes$

The tensor product of abelian groups A, B is the abelian group $A \otimes B$ with generators $a \otimes b$ $a \in A, b \in B$ and relations

$$(a + a') \otimes b = a \otimes b + a' \otimes b$$

$$a \otimes (b + b') = a \otimes b + a \otimes b'$$

Identity: $0 \otimes 0 = 0 \otimes b = a \otimes 0$

Inverses: $-(a \otimes b) = -a \otimes b = a \otimes -b$.

Universal Property

$$\left\{\begin{matrix}\text{Bilinear maps} \\ \text{from } A\times B\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Homomorphisms} \\ \text{from } A\otimes B\end{matrix}\right\}$$

Basic Properties

(i) $A\otimes B \cong B\otimes A$

(ii) $(\bigoplus A_i)\otimes B \cong \bigoplus (A_i\otimes B)$

(iii) $(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$

(iv) $\mathbb{Z}\otimes A \cong A$

(v) $\mathbb{Z}/n\mathbb{Z}\otimes A \cong A/nA$

(vi) $f: A\to A'$, $g: B\to B'$ $\rightsquigarrow$ $f\otimes g : A\otimes B \to A'\otimes B'$

(vii) $\varphi : A\times B \to C$ bilinear $\rightsquigarrow$ $f: A\otimes B \to C$

Back to Cross Product

Property (vii) $\rightsquigarrow$ homomorphism

$$H^*(X,\mathbb{Z})\otimes H^*(Y,\mathbb{Z}) \longrightarrow H^*(X\times Y,\mathbb{Z})$$
$$a\otimes b \longmapsto a\times b$$

The left hand side has multiplication

$$(a\otimes b)(c\otimes d) = (-1)^{|b||c|}\, ac\otimes bd$$

Check: The above map is a ring homomorphism.

THEOREM. (Künneth) $H^*(X,\mathbb{Z}) \otimes H^*(Y,\mathbb{Z}) \xrightarrow{\text{cross}} H^*(X\times Y, \mathbb{Z})$ ring isomorphism
if $H^*(X,\mathbb{Z})$ or $H^*(Y,\mathbb{Z})$ is fin. gen., free.

Exterior Algebras

$\Lambda[\alpha_1, \alpha_2, \ldots]$ = graded tensor product of the $\Lambda[\alpha_i]$, $|\alpha_i|$ odd
As an abelian group, gen by $\alpha_{i_1} \cdots \alpha_{i_k}$ $\quad i_1 < \cdots < i_k$
Multiplication given by $\alpha_i \alpha_j = -\alpha_j \alpha_i \quad i \neq j$

$$\Rightarrow \alpha_i^2 = 0.$$

Cor: $H^*(T^n, \mathbb{Z}) \cong \Lambda[\alpha_1, \ldots, \alpha_n] \qquad |\alpha_i| = 1.$

$\rightsquigarrow$ elts of H^* are sums of: intersect with oriented coordinate tori

More generally, if X is product of odd-dim spheres
$H^*(X) \cong \Lambda[\alpha_1, \ldots, \alpha_n]$ but $|\alpha_i|$ varies.

For even-dim spheres get $\mathbb{Z}[\alpha]/(\alpha^2)$ factors.

Idea of Proof: Induct on dimension.