Exact Sequences

A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

is <u>exact</u> if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$

<u>chain complex</u> if $\operatorname{im} \alpha_{n+1} \subseteq \ker \alpha_n$ i.e. $\alpha_n \circ \alpha_{n+1} = 0$.

Facts:

(i) $0 \longrightarrow A \xrightarrow{\alpha} B \iff \alpha$ injective

(ii) $A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ surjective

(iii) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0 \iff \alpha$ isomorphism.

(iv) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \iff C \cong B/A$ "short exact sequence".

Four Theorems

① Long exact seq. for collapsing subcomplex.

(1') Long exact seq. for pair

③ Excision

② Mayer-Vietoris.

Collapsing a Subcomplex

Theorem. $(X, A) = $ CW-pair.

① There is an exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}(X/A)$$
$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{q_*} \tilde{H}_{n-1}(X/A) \longrightarrow \cdots$$
$$\cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0.$$

where $i : A \hookrightarrow X$, $q : X \to X/A$.

Cor: $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

Proof: Induction on n.
$\tilde{H}_0(S^0) \cong \mathbb{Z}$ ✓
For $n > 0$: $(X, A) = (D^n, S^{n-1}) \rightsquigarrow X/A \cong S^n$.
By theorem:

$$\cdots \longrightarrow \tilde{H}_i(D^n) \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \tilde{H}_{i-1}(D^n) \longrightarrow \cdots$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$ ∎

To prove the Theorem, will do something more general...

Cor (Brouwer Fixed Pt Thm): Every $f : D^n \to D^n$ has a fixed point.

Proof: If not, exists retraction $r : D^n \to \partial D^n$
Consider $\tilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(\partial D^n)$
composition is id & 0 contradiction. ∎

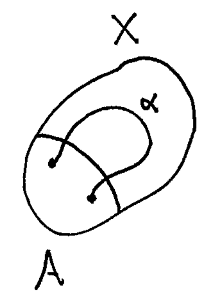
RELATIVE HOMOLOGY

$A \subseteq X \rightsquigarrow C_n(X,A) \cong C_n(X)/C_n(A)$

Since ∂ takes $C_n(A)$ to $C_{n-1}(A)$, have chain complex

$$\cdots \to C_n(X,A) \to C_{n-1}(X,A) \to \cdots$$

$\rightsquigarrow$ relative homology groups $H_n(X,A)$.



Elements of $H_n(X,A)$ are rep by relative cycles:

$$\alpha \in C_n(X) \quad \text{s.t. } \partial\alpha \in C_{n-1}(A)$$

A relative cycle is trivial in $H_n(X,A)$ iff it is a relative boundary:

$$\alpha \in C_n(X) \quad \alpha = \partial\beta + \gamma \quad \text{some } \beta \in C_{n+1}(X), \gamma \in C_n(A)$$

Will show: $H_n(X,A) \cong H_n(X/A)$.

Goal: Long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A)$$

$$\to H_{n-1}(A)$$

Proof is "diagram chasing".

To start:

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_*"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n(X) \arrow[r, "q_*"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_*"] & C_{n-1}(X) \arrow[r, "q_*"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow$ Short exact sequence of chain complexes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d, "i"] & C_n(A) \arrow[r] \arrow[d] & C_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "q"] & C_n(X) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X,A) \arrow[r] \arrow[d] \arrow[uur, dashed, bend right] & C_n(X,A) \arrow[r] \arrow[d] \arrow[uur, dashed, bend right] & C_{n-1}(X,A) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Commutativity of squares $\Rightarrow$ i_*, q_* chain maps

$\rightsquigarrow$ induced maps on homology.

Need to define $\partial : H_n(X,A) \to H_{n-1}(A)$

Let $c \in C_n(X,A)$ a cycle.

$c = q(\tilde{c})$ $\quad \tilde{c} \in C_n(X)$

$\partial\tilde{c} \in \ker q$ by commutativity.

$\Rightarrow$ $\tilde{c} = i(a)$ some $a \in C_{n-1}(A)$ by exactness.

and $\partial a = 0$ by commut: $i\partial(a) = \partial i(a) = \partial\partial(\tilde{c}) = 0$.

i inj.

Set $\partial[c] = [a] \in H_{n-1}(A)$.

Claim: $\partial : H_n(X,A) \to H_{n-1}(A)$ is a well-defined homomorphism.

Well-defined:
- a determined by $\partial\tilde{c}$ since i injective
- different choice $\tilde{c}'$ for $\tilde{c}$ would have
  $\tilde{c}' - \tilde{c} \in C_n(A)$ i.e. $\tilde{c}' = \tilde{c} + i(a')$
  $\Rightarrow a$ changes to $a + \partial a'$
  since $i(a+\partial a') = i(a) + i(\partial a') = \partial\tilde{c} + \partial i(a') = \partial(\tilde{c} + i(a'))$
- different choice for c in $[c]$ is of form $c + \partial c'$
  $c' = q(\tilde{c}')$ some $\tilde{c}' \rightsquigarrow c + \partial c' = c + \partial q(\tilde{c}')$
  $= c + q(\partial\tilde{c}') = q(\tilde{c} + \partial\tilde{c}')$
  so $\tilde{c}$ replaced by $\tilde{c} + \partial\tilde{c}' \rightsquigarrow \partial\tilde{c}$ unchanged.

Homomorphism: Say $\partial c_1 = a_1$, $\partial c_2 = a_2$ via $\tilde{c}_1, \tilde{c}_2$
Then $q(\tilde{c}_1 + \tilde{c}_2) = c_1 + c_2$
$i(a_1 + a_2) = \partial(\tilde{c}_1 + \tilde{c}_2)$
so $\partial(c_1 + c_2) = a_1 + a_2$.

Theorem. The following sequence is exact:
(1')
$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Proof: More diagram chasing. We'll do 2 of the 6 inclusions needed.

$\operatorname{Im}\partial \subseteq \ker i_*$ i.e. $i_*\partial = 0$
$i_*\partial$ takes $[c]$ to $[\partial\tilde{c}] = 0$.

$\ker i_* \subseteq \operatorname{Im}\partial$: Say $a \in C_{n-1}(A)$, $a \in \ker i_* \Rightarrow i(a) = \partial b$ $b \in C_n(X)$
$\Rightarrow q(b)$ a cycle since $\partial q b = q\partial b = q i(a) = 0$
& ∂ takes $[q(b)]$ to $[a]$ ∎

Some facts about relative homology.

Prop: $H_n(X,A)=0 \ \forall n \iff H_n(A)=H_n(X) \ \forall n$.

Can define reduced relative homology

$$\rightsquigarrow \tilde{H}_n(X,A) = H_n(X,A) \text{ whenever } A \neq \emptyset.$$

Prop: If $f,g:(X,A)\to(Y,B)$ are homotopic through maps of pairs then $f_* = g_*$.

For triples $B \subseteq A \subseteq X$, have

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

and so:

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

Then spectral sequences.

Mayer-Vietoris

<u>Theorem</u> $A, B \subseteq X$ interiors cover X. There is long exact seq:
(2)

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

$$x \mapsto x \oplus -x$$

$$x \oplus y \longmapsto x - y$$

$$x = x_A + x_B \mapsto \partial x_A$$

- Reduced version formally identical.
- Mayer-Vietoris is abelian version of Van Kampen: For $A \cap B$ path conn

$$MV \Longrightarrow H_1(X) = H_1(A) \oplus H_1(B) / H_1(A \cap B)$$

<u>Examples</u> ① $X = S^n$ $A, B =$ (neighborhoods of) hemispheres

$$\tilde{H}_i(A) \oplus \tilde{H}_i(B) = 0 \quad \forall i.$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

② $X =$ Klein bottle $A, B =$ (nbhds of) Möbius bands

$$A, B, A \cap B \simeq S^1 \rightsquigarrow$$

$$0 \to H_2(X) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(K) \to 0$$

$$1 \mapsto 2 \oplus -2$$

$$\Rightarrow H_2(K) = 0$$

$$H_1(K) \cong H_1(A) \oplus H_1(B) / H_1(A \cap B) = (1,0) \oplus (1,1) / (-2,2)$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Excision

Theorem. ③ Let $Z \subseteq A \subseteq X$ closure $Z \subseteq$ interior A

Then $(X - \cancel{A}^{Z}, A - Z) \hookrightarrow (X, A)$

induces an isomorphism on homology.

Equivalently: $A, B \subseteq X$, interiors cover X.

$(B, A \cap B) \hookrightarrow (X, A)$ induces $\cong$ on H_*

translation $B = X - Z$, $Z = X - B$.

Application: Invariance of ~~Domain~~ Dimension

Theorem: If nonempty open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ are homeomorphic, then $m = n$.

Proof: Let $x \in U$. $H_k(U, U - x) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - x)$ by Excision.

Long exact seq. for $(\mathbb{R}^m, \mathbb{R}^m - x)$:

$$\cdots \to H_k(\mathbb{R}^m) \to H_k(\mathbb{R}^m, \mathbb{R}^m - x) \to H_{k-1}(\mathbb{R}^m - x) \to H_{k-1}(\mathbb{R}^m) \to \cdots$$

$$\Longrightarrow H_k(\mathbb{R}^m, \mathbb{R}^m - x) \cong H_{k-1}(\mathbb{R}^m - x)$$

But $H_{k-1}(\mathbb{R}^m - x) \cong H_{k-1}(S^{m-1})$ since $\mathbb{R}^m - x$ def. ret to S^{m-1}

Thus:

$$H_k(U, U - x) = \begin{cases} \mathbb{Z} & k = m \\ 0 & \text{o.w.} \end{cases}$$

In other words, can detect m from homology groups. $\blacksquare$

Excision also used to show $H_n(X, A) \cong \tilde{H}_n(X/A)$, so Theorem 2 implies Theorem 1. See Hatcher Prop 2.22

Remains to prove Excision and Mayer-Vietoris.

Idea: Subdivide.

Another homology: X = space

$\mathcal{U} = \{U_j\}$ collection of subspaces whose interiors cover X.

$C_n^{\mathcal{U}}(X)$ = chains $\sum n_i \sigma_i$ so each σ_i has image in some U_j

$\partial(C_n^{\mathcal{U}}(X)) \subseteq C_{n-1}^{\mathcal{U}}(X) \rightsquigarrow$ chain complex

$\rightsquigarrow H_n^{\mathcal{U}}(X)$

<u>Prop</u>: $H_n^{\mathcal{U}}(X) \cong H_n(X)$

Specifically, there is a subdivision operator $\rho: C_n(X) \to C_n^{\mathcal{U}}(X)$ that is a chain homotopy inverse to $\iota: C_n^{\mathcal{U}}(X) \to C_n(X)$.

<u>Proof</u> <u>of</u> <u>Excision</u>. To show $H_n(B, A\cap B) \cong H_n(X,A)$.

Let $\mathcal{U} = \{A, B\}$

Note $C_n^{\mathcal{U}}(A)$ naturally identified with $C_n(A)$. by ρ and ι.

$$\Longrightarrow \quad C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A) \longrightarrow C_n(X)/C_n(A)$$

induces isomorphism $H_n^{\mathcal{U}}(X,A) \cong H_n(X,A)$.

But: $$C_n(B)/C_n(A\cap B) \longrightarrow C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$$

obviously an isomorphism: both are free on simplices lying in B but not A. So $H_n(B, A\cap B) \cong H_n^{\mathcal{U}}(X,A)$.

$\blacksquare$

__Proof of Mayer-Vietoris.__ Recall $X = A \cup B$.

Let $\mathcal{U} = \{A, B\}$

There is a short exact seq. of chain complexes:

$$0 \longrightarrow C_n(A\cap B) \longrightarrow C_n(A) \oplus C_n(B) \longrightarrow C_n^{\mathcal{U}}(X) \longrightarrow 0$$

$$x \longmapsto x \oplus -x$$

$$x \oplus y \longmapsto x+y$$

$\rightsquigarrow$ long exact seq. in homology as before.

Substituting $H_n(X)$ for $H_n^{\mathcal{U}}(X)$ (Proposition)

$\rightsquigarrow$ Mayer-Vietoris sequence. ▨

A description of $\partial : H_n(X) \longrightarrow H_{n-1}(A\cap B)$:

$\alpha \in H_n(X)$ rep. by cycle z

$z = x + y \quad x \in C_n(A),\ y \in C_n(B)$

$\partial x = -\partial y$ since $\partial z = 0$.

Set $\partial\alpha = \partial x$.

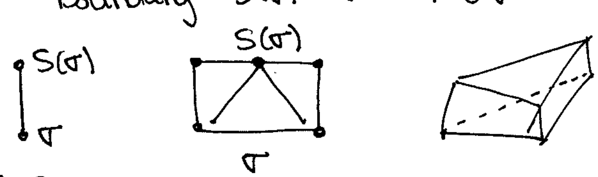
__Proof of Prop.__ Let S = barycentric subdivision.

First show S is a chain homotopy equiv.

then take $\rho = S^N$.

Want $T : C_n(X) \longrightarrow C_{n+1}(X)$ s.t. $T\partial + \partial T = S - id$.

i.e. for any n-simplex σ want $(n+1)$-chain $T\sigma$ with boundary $S(\sigma) - \sigma - T\partial\sigma$



$n=0$ $\quad$ $n=1$ $\quad$ $n=2$

Do $n=1$ case on all 3 sides. Then join all simplices to barycenter on top.

Homology and Fundamental Group

In many examples, can see $H_1(X) = \pi_1(X)^{ab}$,
e.g. surfaces, $S^1 \vee S^1$, S^n

<u>Theorem</u>. $H_1(X) = \pi_1(X)^{ab}$

<u>Proof</u>. Regarding loops as 1-cycles, there is a map

$$h : \pi_1(X) \longrightarrow H_1(X)$$

To show h a well-defined, surjective homomorphism with kernel $[\pi_1(X), \pi_1(X)]$

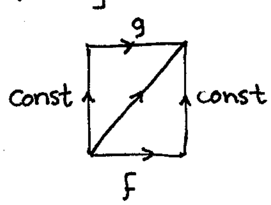
Write $\simeq$ for homotopy, $\sim$ for homology.

Fact 1. const ~ 0

Pf. $H_1(pt) = 0$ also: const loop $= \partial$ const. 2-simplex

Fact 2. $f \simeq g \Rightarrow f \sim g$

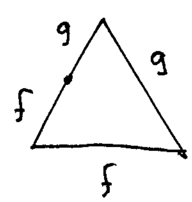
Pf.



boundary $= f - g$

Fact 3. $f \cdot g \sim f + g$

Pf.



boundary $= g - f\cdot g + f$

Fact 4. $\bar{f} \sim -f$

Pf. $f + \bar{f} \overset{③}{\sim} f \cdot \bar{f} \overset{②}{\sim} \text{const} \overset{①}{\sim} 0$

Well-defined. Facts 2 and 3.

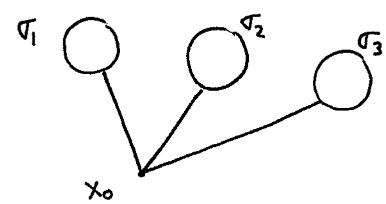
Surjective. Let $\sum n_i \sigma_i$ = 1-cycle

Relabel. $\sum \pm \sigma_i$

By Fact 4, rewrite as $\sum \sigma_i$

Use Fact 3 to organize into loops, relabel $\sum \sigma_i$

Use Facts 3 and 4 to combine into one loop σ:



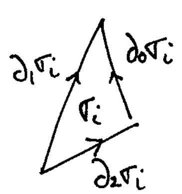
The loop σ is in image of h.

Note $[\pi_1(X), \pi_1(X)] \subseteq \operatorname{Ker} h$ since $H_1(X)$ abelian.

So say $h(f) \sim 0$. To show $f \in [\pi_1(X), \pi_1(X)]$, i.e. $f = 0$ in $\pi_1(X)^{ab}$.

$$h(f) \sim 0 \Rightarrow f = \partial\left(\sum \sigma_i\right) \qquad \sigma_i = \text{2-simplex (singular)}$$

$$= \sum (\partial_0 \sigma_i - \partial_1 \sigma_i + \partial_2 \sigma_i)$$



Modify all σ_i by homotopy so all vertices map to basepoint for $\pi_1(X) \Rightarrow$ Can regard the sum in $\pi_1(X)^{ab}$

In $\pi_1(X)$ have $(\partial_2\sigma_i) \cdot (\partial_0 \sigma_i) = (\partial_1 \sigma_i)$ see picture

$\Rightarrow$ each term of sum is 0 in $\pi_1(X)^{ab}$ ∎

Alternate ending. Want to show

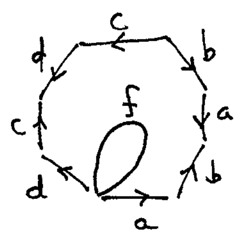
$$h(f) = 0 \Rightarrow f \in [\pi_1(X), \pi_1(X)]$$

$$h(f) = 0 \Rightarrow f = \partial \sum \sigma_i$$

Claim: $\sum \sigma_i$ represents an orientable surface with one boundary, namely f.

Pf: Adjacent triangles must have both ∂'s clockwise or both counterclockwise.

Classification of surfaces $\Rightarrow$ $\sum \sigma_i$ is



$\Rightarrow$ f a product of g commutators. ▨

Some History

An *n-manifold* is a Hausdorff space where each point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Poincaré's First Conjecture. If X is a 3-manifold with $H_1(X) = 0$, then X is homeomorphic to S^3.

Counterexample: Poincaré Dodecahedral Space.

Take a solid dodecahedron, glue opposite faces with $2\pi/10$ clockwise twist. This has same homology as S^3 ("homology sphere")

This led Poincaré to develop π_1. $\rightsquigarrow |\pi_1(PDS)| = 120$.

The last theorem shows π_1 has more information than H_1. Sometimes this is important information!