Cup, Cap, and Poincaré Duality

Poincaré duality.
$$H^k(X) \xrightarrow{\cong} H_{n-k}(X)$$
$$\varphi \longmapsto [M] \frown \varphi$$

Also. Under this isomorphism, cup product corresponds to intersection: $\varphi \cup \psi \longmapsto \varphi^* \cap \psi^*$

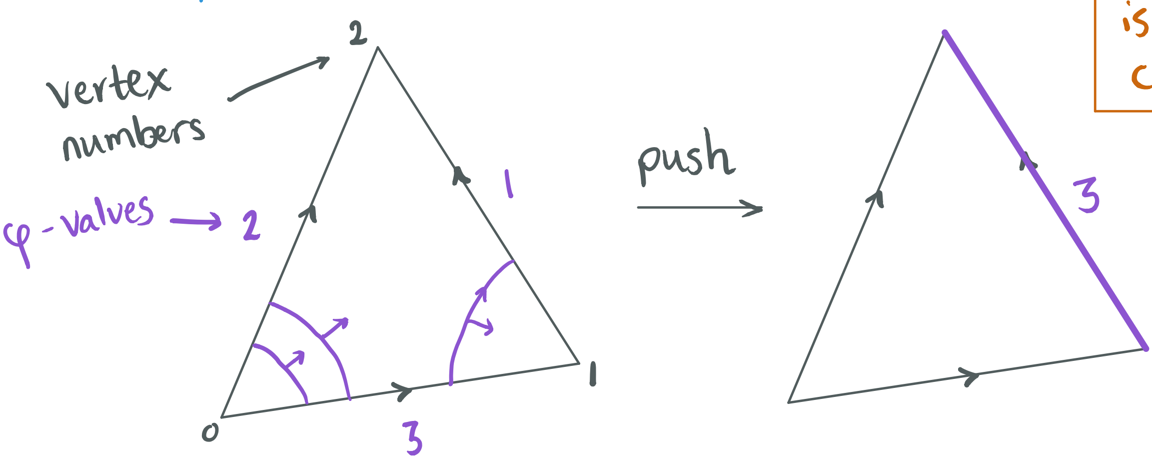
We'll work with Δ-complexes, simplicial (co)homology.

Cap

Idea. Realize cohomology class φ as "intersect with dual object." Push dual in each simplex toward highest vertex (this is well-defined across different simplices in a Δ-complex).
Result is $[M] \frown \varphi = \varphi^*$

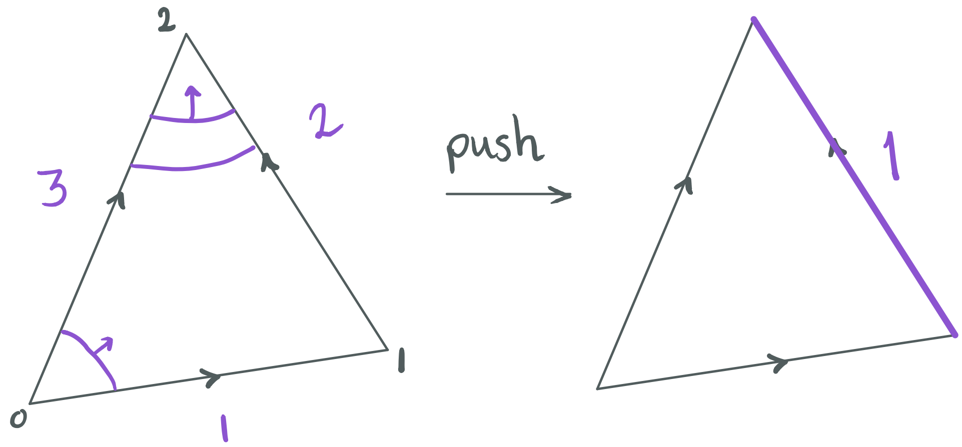
Example 1. $n = 2$, $k = 1$

Note: in a manifold, orientation is same as co-orientation

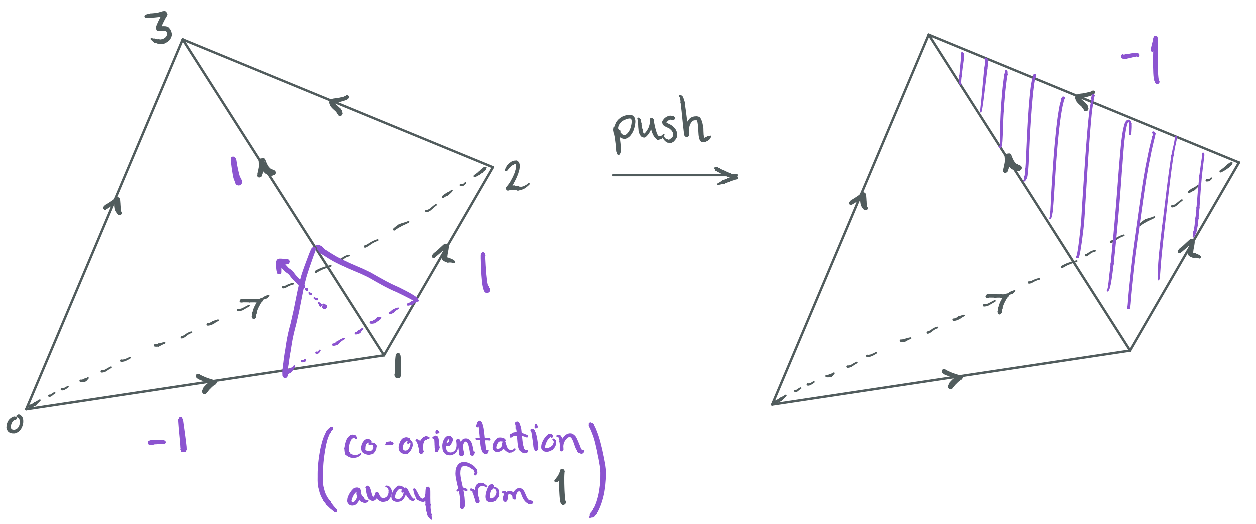


This is exactly what $\varphi \frown [M]$ gives!

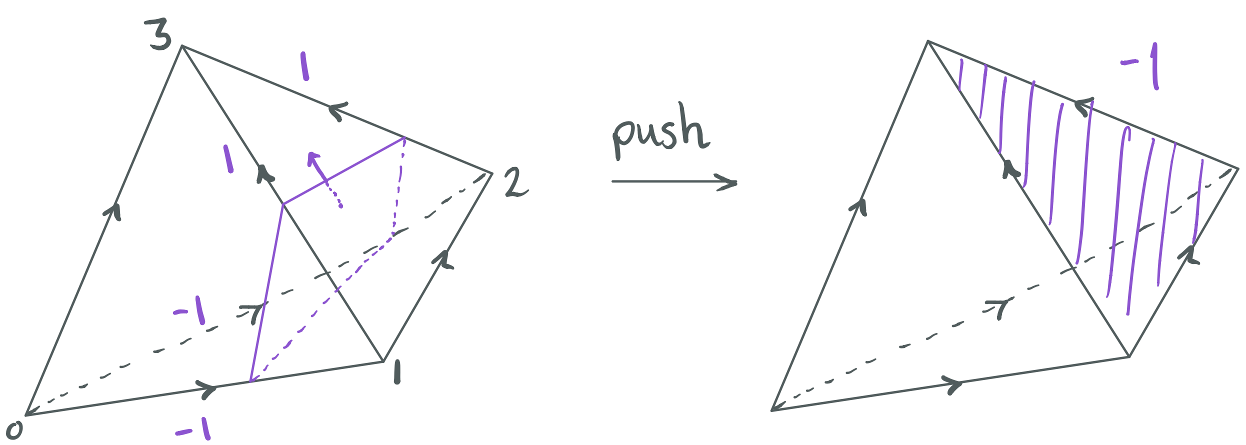
Example 2. $n=2,\ k=1$



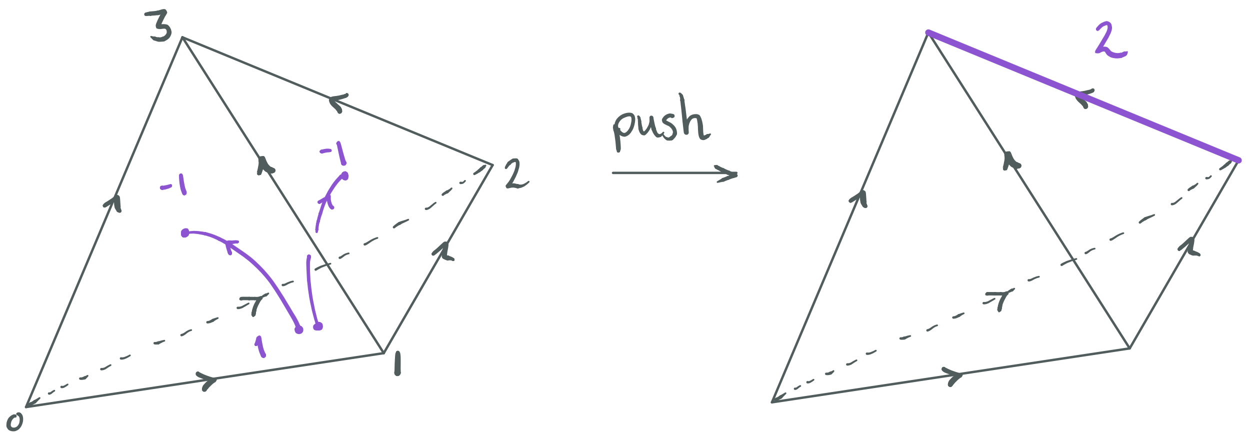
Example 3 $n=3,\ k=1$



Example 4 $n=3,\ k=1$



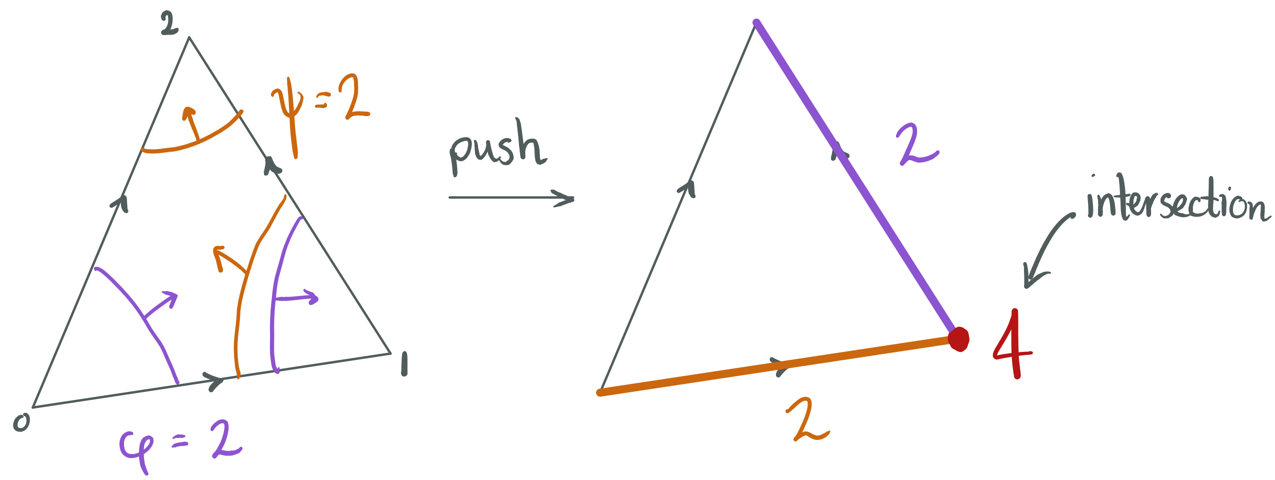
Example 5 $n=3,\ k=2$



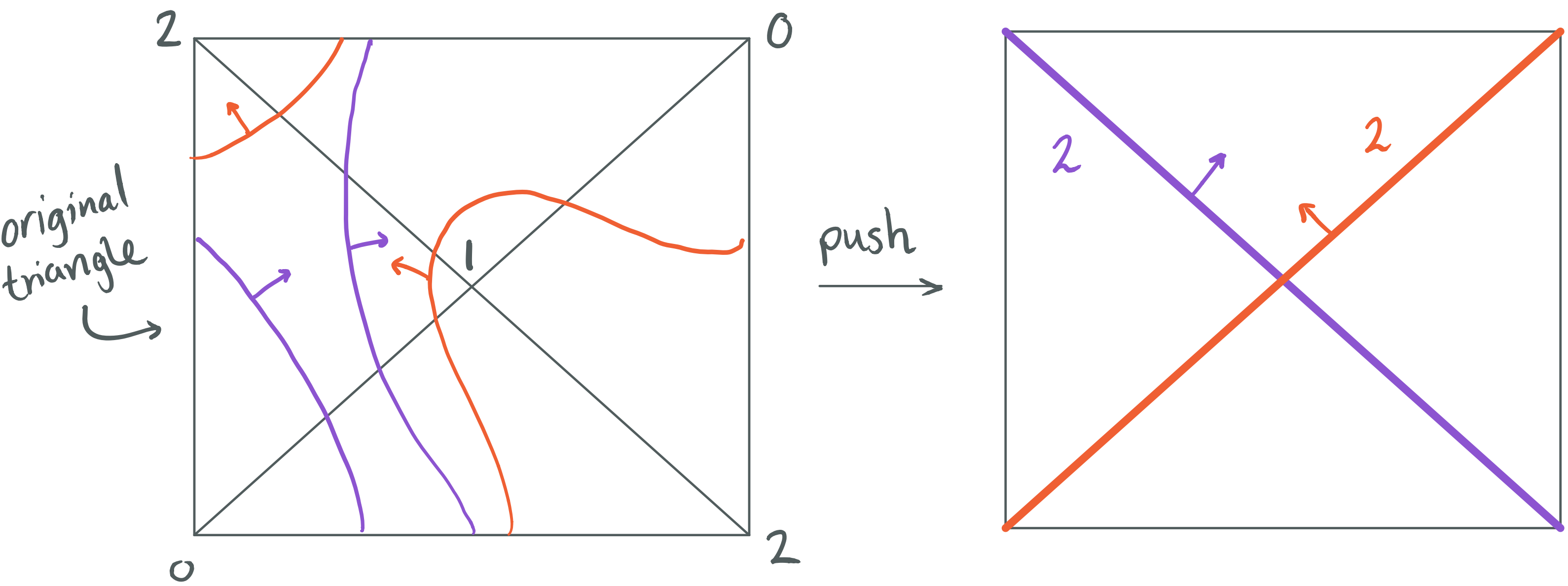
Cup

Idea. To find $\varphi \cup \psi$, push φ up, push ψ down and intersect

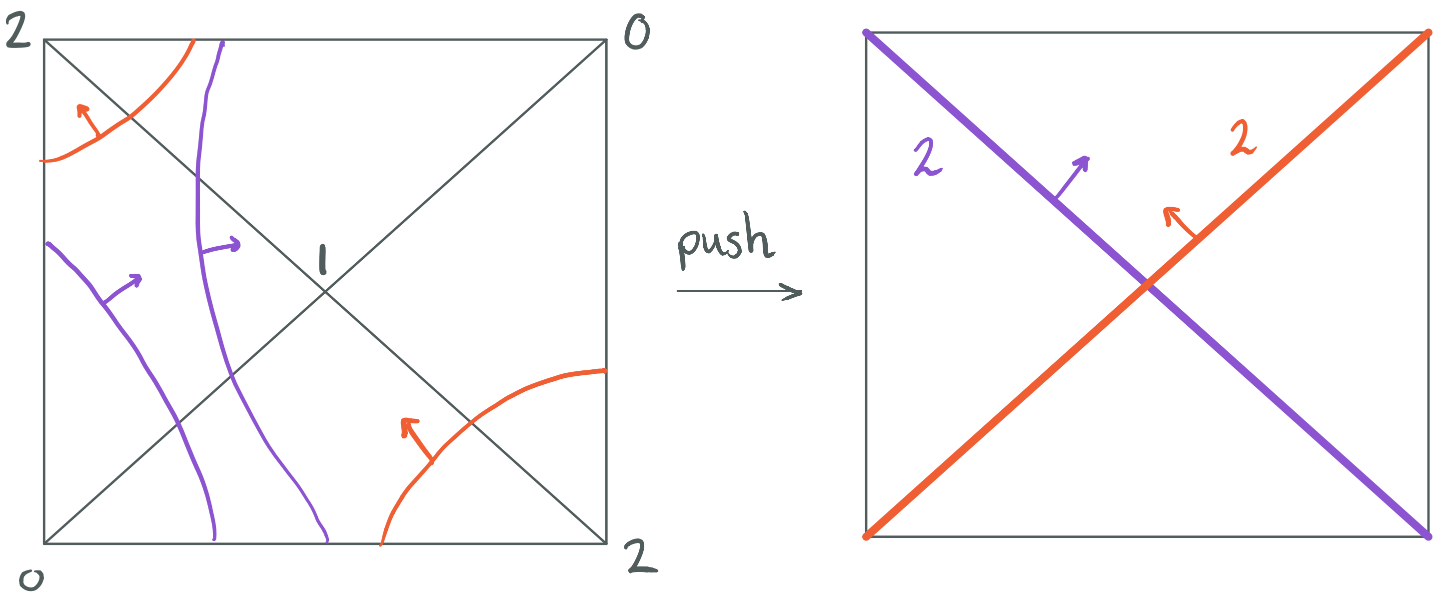
Example 1. $n=2,\ k=1$



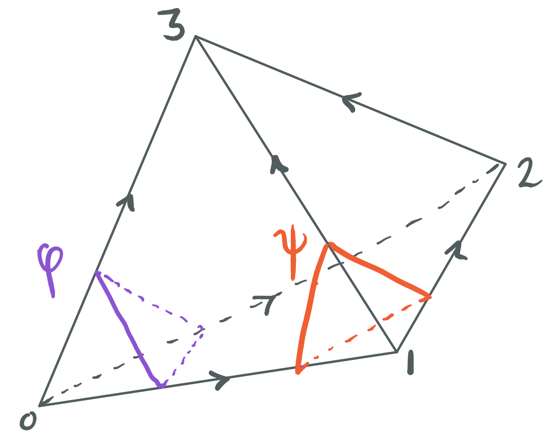
Can view same example in context of nearby triangles:



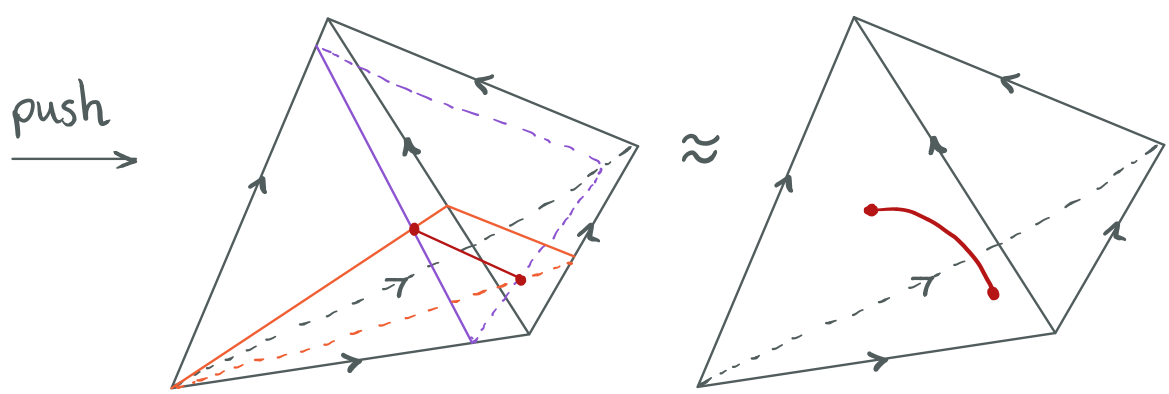
We can modify the curves by homotopy, giving cohomologous cochains:



Example 2. $n=3$, $k=1,1$ (mod 2 this time)



Have $\varphi \cup \psi \in H^2 \rightsquigarrow$ should be dual to a 1-cell. If we push all the way and intersect, get a point (not what we want). If we push <u>almost</u> all the way, we get what we want:



This is dual to $\varphi \cup \psi$!

Note: In the earlier examples, pushing almost all the way also works.