This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

Please send all submissions to the Mathematical Entertainments Editor, **Alexander Shen,** Institute for Problems of Information Transmission, Ermolovoi 19, K-51 Moscow GSP-4, 101447 Russia; e-mail:shen@landau.ac.ru

An Unfair Game

In this game, our probabilistic intuition (at least mine) fails drastically. I learned it from A. L. Brudno.

Alice and Bob toss a fair coin. Before they start, each of them selects a three-bit string, like 010 or 111. A player wins when these bits appear (as consecutive bits), assuming that the opponent has not won earlier. Bob politely suggests Alice choose her string first. As he explains, it gives her more freedom: she may choose any string she wishes, and he has to restrict himself to the remaining seven strings (it is not allowed to have the same string for both players for evident reasons).

Should Alice believe him? Stop reading here and think a minute about this game. Unless you do the computations, you may be surprised to know how biased the game is. In fact, Bob's chances are (at least) twice as good if he is clever enough. For example, if A chooses 010, then B can choose 001 and win with probability 2/3.

Let us check that. p_{00}, p_{01}, p_{10} and p_{11} denote the probabilities that A wins, assuming that the last two bits are given and nobody has won earlier. The equations are

$$\begin{aligned}p_{00} &= p_{00}/2,\\ p_{01} &= 1/2 + p_{11}/2,\\ p_{10} &= p_{00}/2 + p_{01}/2,\\ p_{11} &= p_{10}/2 + p_{11}/2.\end{aligned}$$

For example, the second equation says that after . . . 01, the two events 0 (then A wins for sure) and 1 (then A wins with probability p_{11}) are equally probable. The first equation implies that $p_{00} = 0$, and it is easy to see why it is. Indeed, after 00, the only chance for Alice is never get 1, and this event has zero probability.

The only solution for this system of linear equations is $p_{00} = 0$, $p_{01} = 2/3$, $p_{10} = p_{11} = 1/3$. The average of these four probabilities (that is, the probability for A to win the game from the beginning) is 1/3.

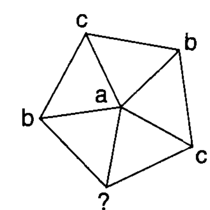
Some other strings are even worse for her. For example, string 000 wins against 100 only with probability 1/7. (If Alice has no luck in the first three rounds, she cannot win at all!)

Colorings and Coverings

We continue our collection of problems with unexpected solutions.

Consider a convex polyhedron whose faces are all triangles. We want to color its vertices in three colors such that each triangle has vertices of three different colors.

An evident necessary condition is that each vertex is incident to an even number of faces: the colors around a vertex alternate, and if the number of faces is odd, there is no color for the last vertex (Fig. 1).



Problem: Prove that this condition is also sufficient.

It is a well-known problem but I do not know its origin. The following solution was suggested by Maxim Kontsevich.

Consider six copies of each triangle face of the polyhedron; each copy carries one of the six possible colorings of three its vertices. Now we glue the neighboring triangles together, taking into account the coloring: two copies are glued only if both common vertices have the same color. (This procedure may encounter difficulties if done in three-dimensional space, since different layers intersect each other.)

Let us look what happens around any vertex. If the number of adjacent triangles is odd, we get three two-sheet coverings (each similar to the Riemann surface for $\sqrt{z}$ near the point $z = 0$). However, if the number is even, we get a regular covering with six sheets. But the surface of the polyhedron is a topological sphere, so its fundamental group is trivial, and this covering is split into six copies of the polyhedron, each properly colored. That's all!

For the torus, the situation is different: some triangulations allow 3-colorings whereas others don't. Indeed, if we color vertices along a fundamental cycle, at the end we may get a coloring which is inconsistent with the initial one. In this case the triangulation does not allow 3-coloring. And if this does not happen for either fundamental cycle, a 3-coloring exists.

Tilings and Polyhedra Revisited

Since the first 1997 issue of *The Mathematical Intelligencer* appeared, I've received many comments about the Entertainments column, and some need answering.

First, about the cube and tetrahedron of equal volume that cannot be split into equal parts. Several readers, including C. Kenneth Fan, Peter Freud, John Stillwell, and Wim Veldman, made important comments: I did not mention that it is a famous Hilbert problem [included in his list of most important mathematical problems (1900)] and was solved by M. Dehn (1900). The solution I explained is due to Hadwiger (1948). This information can be found in the survey written by V. G. Boltianskii, who also wrote a popular book on the subject (*Hilbert's Third Problem*, Washington, 1978).

Another important remark is about the use of the axiom of choice. Let me quote a letter from Pierre Deligne:

> . . . the use of Dehn's invariant to show that one cannot go from a cube to a regular tetrahedron by cutting and pasting does not in fact require the axiom of choice. The argument gives a cut and paste invariant in
>
> $$D := \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/(2\pi\mathbb{Z}),$$
>
> which is 0 for the cube and of the form $\ell \otimes \alpha$ for the tetrahedron, where α is the dihedral angle and $\ell \neq 0$. If you wanted to show the existence of a morphism m from D to $\mathbb{Q}$— or $\mathbb{R}$— with $m(\ell \otimes \alpha) \neq 0$, I expect you would indeed need the axiom of choice. However, you only need to prove that $\ell \otimes \alpha \neq 0$. For this, it suffices to check that $\alpha \notin 2\pi\mathbb{Q}/2\pi\mathbb{Z}$. Indeed, $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$ is, essentially by definition, the inductive limit of the $X \otimes Y$ for $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}/2\pi\mathbb{Z}$ finitely generated subgroups. It hence suffices to check that for all finitely generated X and Y such that $\ell \in X$ and $\alpha \in Y$, $\ell \otimes \alpha$ in $X \otimes Y$ is not 0. This does not require the axiom of choice! Otherwise said: if a cutting and pasting transforms the cube into the tetrahedron, it will involve only finitely many lengths and angles. Let X and Y be the subgroup of $\mathbb{R}$ (resp., $\mathbb{R}/\mathbb{Z}$) generated by those lengths (resp., angles). Then Dehn's proof gives $\ell \otimes \alpha = 0$ in $X \otimes Y$.
>
> . . . The axiom of choice is not used, but as I presented the argument, *tertium non datur* is: to claim that a finitely generated subgroup X of $\mathbb{R}$ is isomorphic to $\mathbb{Z}n$ for some n, or for the resulting claim that, if $\ell \in X$, there is m: $X \to \mathbb{Z}$ with $m(\ell) \neq 0$. This makes the proof nonconstructive: as often, it is a use of *tertium non datur*, rather than a use of the axiom of choice, which threatens constructivity. Here, the remedy is simple: define X not as a subgroup of $\mathbb{R}$ but as abstractly generated by the segments occurring in a cutting and pasting, and related by the relations among length of segments needed. Otherwise said: the invariant lives in the inductive limit of the $X \otimes Y$, for X (resp., Y) in the filtering category of $\mathbb{Z}$-modules effectively finitely presented (cokernel of some map $\mathbb{Z}p \to \mathbb{Z}q$), given with a map to $\mathbb{R}$ (resp., $\mathbb{R}/\mathbb{Z}$).

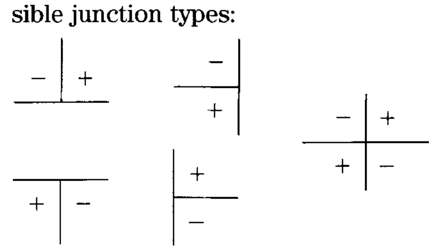
Now about the tilings problem (a rectangle that can be tiled by rectangles each having at least one integer side, has an integer side). It is a shame that I didn't know about the excellent article by Stan Wagon "Fourteen proofs of a result about tiling a rectangle" [*American Mathematical Monthly* 94 (1987), 601–617]. Several readers told me of this article, and I strongly recommend it for the nice proofs explained in a very clear way. (Some of them were independently found by column readers!) The already cited letter from P. Deligne gives one more:

> . . . A similar remark applies to the semi-integral rectangles you consider in the first part of the article. Here, to a rectangle A with sides parallel to the axes: $A = [x_0, x_1] \times [y_0, y_1]$, one attaches
>
> $$c(A) = (\delta(x_1) - \delta(x_0)) \otimes (\delta(y_1) - \delta(y_0))$$
>
> in $\mathbb{Z}^{(\mathbb{R}/\mathbb{Z})} \otimes \mathbb{Z}^{(\mathbb{R}/\mathbb{Z})}$. This invariant has the virtues that (i) it vanishes if $x_1 - x_0$ or $y_1 - y_0$ is integral, i.e., if A is semi-integral, (ii) if a rectangle A is dissected into rectangles A_i, then $c(A) = \sum c(A_i)$. In addition, the converse of (i) is true. As before, it suffices to check this for $\mathbb{R}/\mathbb{Z}$ replaced by a finite subset, in which case it is trivial. Nothing more is required for your proof.
>
> By the way, this version of the argument works as well with $\mathbb{Z}$ replaced by any subgroup of $\mathbb{R}$ (and a different subgroup for each axis is allowed).

Let me repeat this nice proof without mentioning tensor products explicitly. For any rectangle R consider its "imprint" $i(R)$, defined as a formal linear combination of its four vertices: two opposite corners (northeast and southwest) are taken with plus signs, the two others with minus signs. If a rectangle R is tiled by smaller rectangles $R_1, \ldots, R_n$, then

$$i(R) = i(R_1) + \cdots + i(R_n).$$

This can be seen by looking at all possible junction types:



Now we identify points whose coordinates differ by integers

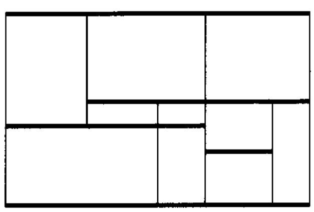
$$(x, y) \sim (x + m, y + n)$$

for any integers m and n. After that, the imprint of any semi-integer rectangle becomes zero (plus vertices are canceled out by minus vertices). Therefore, if all R_i are semi-integer rectangles, their imprints vanish and the imprint of the rectangle R should vanish also. But this means that it is a semi-integer rectangle, too. (End of proof.)

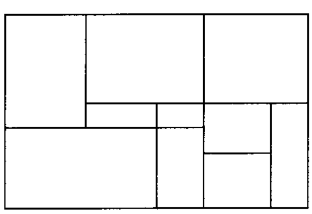
Let me mention two other problems connected with tilings. The first is well known: if a unit square is tiled by squares, their sides are rationals. (The solution I know involves linear algebra and even a bit of mathematical logic!)

The other was submitted by David Gale:

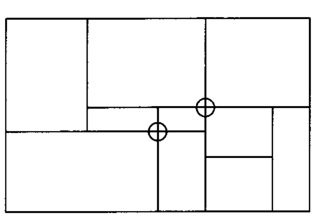
Another Euler-type equation. In a *tiling* of a rectangle by rectangles, the horizontal (vertical) edges of the tiles are partitioned into disjoint horizontal (vertical) *segments*, as shown below. A *cross* is a vertex which is common to four tiles:



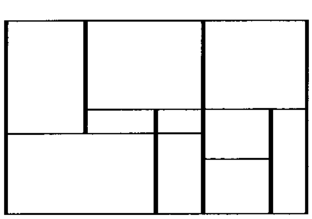
5 Horizontal Segments



10 Tiles



2 Crosses



6 Vertical Segments

Prove that in any such tiling

$$\#\text{Segments} - \#\text{Tiles} + \#\text{Crosses} = 3.$$