# Math 6441

See web page from my web page (Dan Margalit)

Please wear masks.

Grading: HW 60%

Midterm 20%

Final Project 20%

Book: Hatcher.

Recordings: Math 6441 → Lectures → Files → Recordings

Office Hours: TBA.

# Math 6441 : Alg Top

Jan 10

## What is alg. top?

$$\boxed{\text{Space}} \longrightarrow \boxed{\text{Group}}$$

hard easier

$X \rightleftarrows \pi_1(X)$ fundamental gp.

$$\text{abelian}\begin{cases} X \longrightarrow H_k(X) & \text{k-th homology group} \\ X \longrightarrow H^k(X) & \text{k-th cohomology gp.} \end{cases}$$

## What kinds of questions does it answer?

① When are two spaces the same?

e.g. $\mathbb{R}^n \not\cong \mathbb{R}^m$ $m \neq n$.

or $\mathbb{R}^3 -$  $\not\cong \mathbb{R}^3 -$ 

② Embeddings

e.g. Klein bottle $\not\hookrightarrow \mathbb{R}^3$

More general:

What is smallest $N$ so a given manifold embeds in $\mathbb{R}^N$.

Unsolved for $\mathbb{RP}^n$.

## ③ Fixed pt theorems

Brouwer fixed pt thm:

every contin. $D^2 \to D^2$ has a fixed pt.

Borsuk-Ulam thm.

Any contin $S^2 \to \mathbb{R}^2$ has antipodal pts with some image.

## ④ Actions

Which finite gps act freely on $S^n$? Known in some cases

Note: $\mathbb{Z}/n \curvearrowright S^{2k-1}$ freely.

## ⑤ Sections

What is the largest $k$ s.t. a given manifold admits a continuously varying $k$-plane field?

example: Can't-comb-a-monkey-theorem

## ⑥ Group theory

- Every subgp of a free gp is free.
- $[F_n, F_n]$ is not fin. gen.
- Braid groups are torsion free

## ⑦ Algebra

Fund thm of algebra

$\mathbb{R}^3$ is not a field.

## ⑧ Graph theory.

A convex polyhedron made of triangles is 3-cobrable iff it has an even # of triangles at each vertex (Kontsevich proof).

Also: Robotics
Networks
Data science
Exotic manifolds.

## Outline/Overview

## (I) Fundamental Group

elements: loops at basept $/\sim$

operation: concat.

We'll see:

groups ⟷ spaces

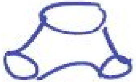
subgps ⟷ covering spaces

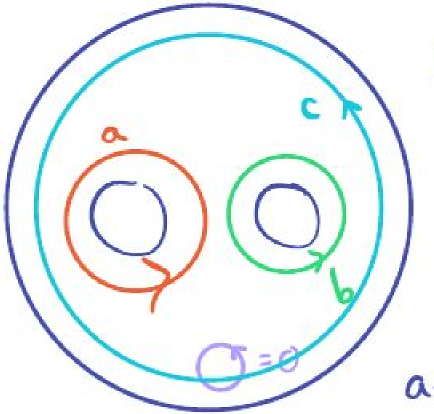
homoms ⟷ maps.

## Ⅱ Homology

Idea:

$H_k(X)$ = abelian gp of k-dim holes in X

X = pair of pants 



$H_1(X) \cong \mathbb{Z}^2$

a+b = c.

## Ⅲ Cohomology

$H^k(X)$ dual to $H_k(X)$

Consists of functions

$$H_k(X) \longrightarrow \mathbb{Z}$$

⇝ ring, not just abel. gp.

Big goal: Poincaré Duality

For $X$ = n-manifold

$$H^k(X) \cong H_{n-k}(X)$$

More precisely, the functions in $H^k(X)$ are: intersect with some elt of $H_k(X)$

What do we mean by space?

Cell complexes a.k.a.

CW complexes

e.g.

Basically: glue cells together

Quotient topology:

$U \subseteq X/\sim$ open iff

its preimage in $X$ open

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We build a CW complex inductively:

(i) Start with a discrete set of pts $X^0$

"0-cells" 0-skeleton

(ii) Inductively form n-skeleton $X^n$ from $X^{n-1}$ by attaching n-cells

n-ball

$D^n_\alpha$ via $\varphi_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}$

index

In torus example:

$X^0 = \bullet$

$X^1 =$ 

$X^2 =$ 

exercise: write $\varphi_\alpha$

Can Either stop at a finite stage, or continue indefinitely.

Topology is the weak topology: a set is open iff its intersection with each cell is open.

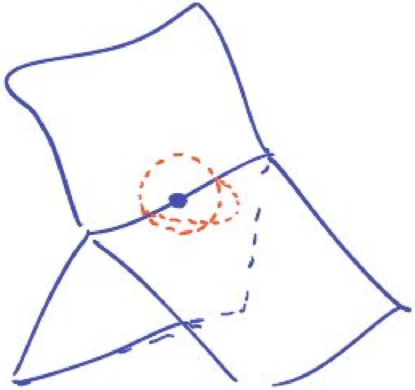
$\dim X = \sup\{\text{dim of cells}\}$

If $\dim X < \infty$

weak top $\equiv$ quotient top.

exercise. Think about continuous paths.

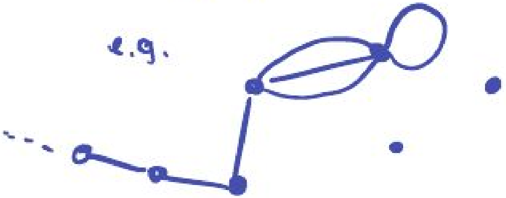
an open set



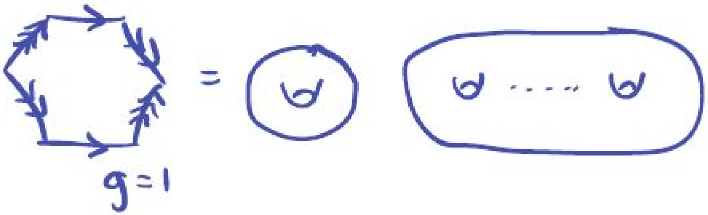
# Examples

① 1-dim CW complexes are graphs

e.g.



② (4g+2)-gon with opp. sides identified



③ $S^n = e^0 \cup e^n$

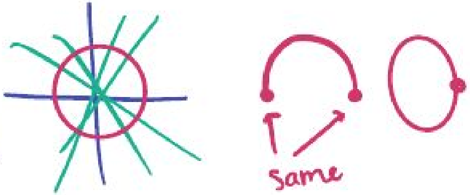
④ $\mathbb{RP}^n$ = space of lines thru 0 in $\mathbb{R}^{n+1}$

$= e^0 \cup e^1 \cup \cdots \cup e^n$.

$\mathbb{RP}^1$ :

||

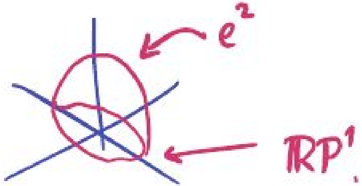
$S^1$/antipod.



$\mathbb{RP}^2$ :

||

$S^2$/antipod



⑤ $\mathbb{CP}^n$ = space of lines thru 0 in $\mathbb{C}^{n+1}$

$= e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$

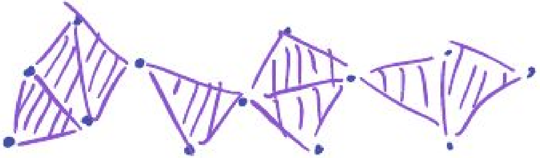
$\mathbb{CP}^1 = \hat{\mathbb{C}} \approx S^2$

Also: can have $n = \infty$ in last 2 examples!

⑥ All smooth closed manifolds

⑦ All top. closed manifolds of dim $\neq 4$ (Manolsecu)

⑧ Networks/Data (Čech complexes)

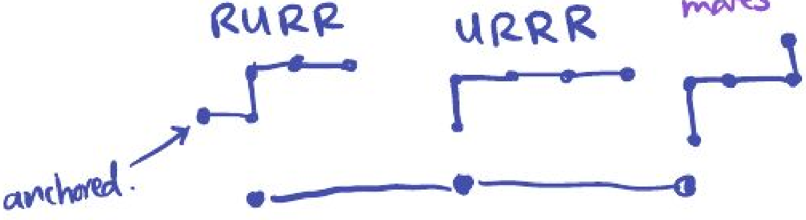


⑨ Robot arms (Ghrist)

(or, seqs of U's & R's)

edges
- swap RU, UR
- change last letter.

square • "commuting"/disjoint moves



exercise: draw the whole thing.

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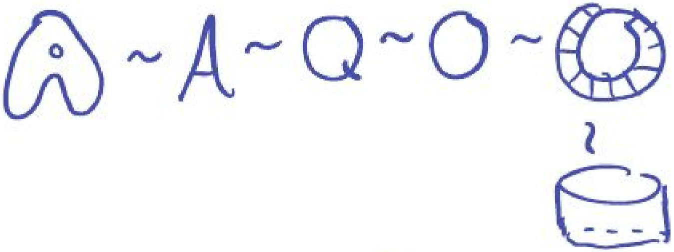
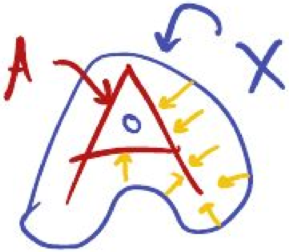
# Equivalence of spaces

Intuition: Two spaces are same if one can be deformed to the other:





Special case. A deformation retraction $X \longrightarrow A$ is a continuous family

$$\{f_t : X \longrightarrow X \mid t \in I\}$$

s.t. $f_0 = id$

$f_1(X) = A$

$f_t|_A = id$ for all $t$.

Continuous means $X \times I \longrightarrow X$

$(x,t) \longmapsto f_t(x)$

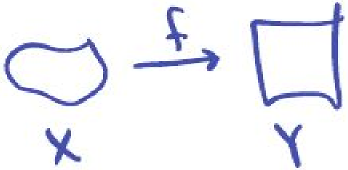
is continuous.

**Example** Given $f: X \to Y$ the **mapping cylinder** is

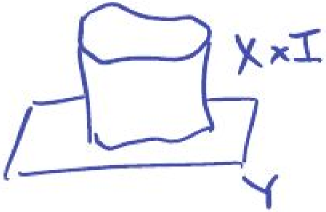
$$M_f = (X \times I) \amalg Y / \sim$$

$$(x,1) \sim f(x)$$

So: Given



Make:



e.g.

is $M_f$ for

$X =$

boundary

$Y = A$

$f: X \to Y$ given by

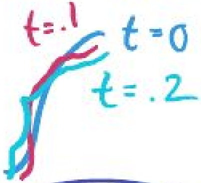
**Fact.** $M_f$ def. retracts to $Y$

# Homotopy equivalence

A **homotopy** is a continuous family $\{f_t : X \to Y : t \in I\}$

**examples** ① def. retr.

②



$X = I$
$Y = \mathbb{R}^2$

③

$X = S^1$
$Y = T$

Say $f_0, f_1$ are **homotopic** maps.

A map $f : X \to Y$ is a **homotopy equivalence** if there is a $g : Y \to X$ s.t.

$$f \circ g \simeq \mathrm{id} \simeq g \circ f$$

↑ homotopic.

Say $X, Y$ homotopy equivalent.

or: same homotopy type.

or: $X \simeq Y$

exercise: This is an equiv. relation.

**Fact.** If $A$ is a def. retr. of $X$ then $A \simeq X$.

Exercise:

 $\simeq$  $\simeq$  $\simeq$ 

Exercise: $\mathbb{R}^n \simeq *$ ← pt.

(def retr.)

Say: $\mathbb{R}^n$ is <u>contractible</u>.

Read: House with two rooms.

# Two Criteria for Homotopy Equivalence

① $(X,A)$ = CW-pair (i.e. $A$ is a subcomplex)

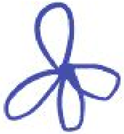
$A$ contractible

$$\Longrightarrow \quad X \simeq X/A \quad \leftarrow \text{identify } A \text{ to one pt.}$$

example. $X$ = graph

$A$ = any edge connecting distinct vertices.

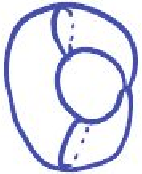
Thus: any graph $\simeq$



example.

$X$ =

$X/A$ =



$X/B$ =

② $(X,A)$ CW-pair.

$f,g : A \to Y$ homotopic

$$\Longrightarrow X \sqcup_f Y \simeq X \sqcup_g Y$$

$$X \sqcup_f Y = (X \amalg Y)/_{a \sim f(a)}$$

# Two Criteria for Hom. Equiv

① $(X, A)$ CW pair

$A$ contractible

$\Rightarrow X \simeq X/A$

② $(X, A)$ CW pair

$f, g : A \to Y$ homotopic

$\Rightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

## Examples

Jan 19
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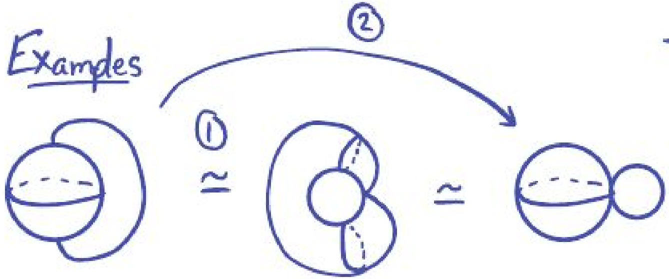
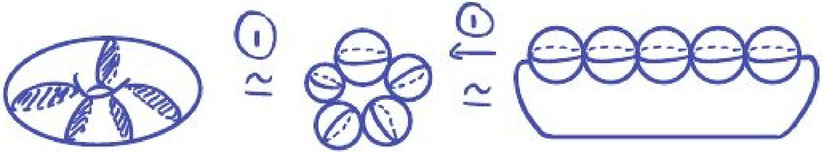
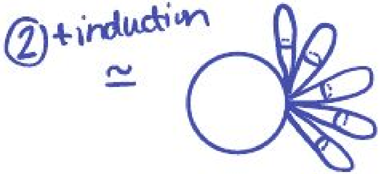
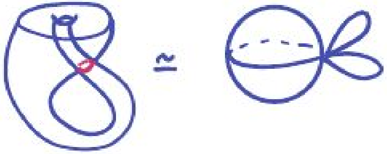






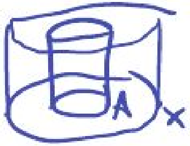
exercise. prove these equivalences using ① & ②

HW:



Both proofs use:

Homotopy extension property



Say (X,A) has HEP

if whenever we have

$f_0 : X \to Y$

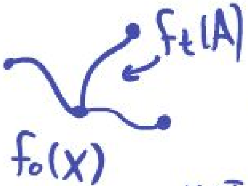
$f_t : A \to Y$

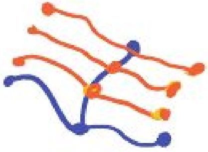
restriction of first $f_0$ to A is the 2nd $f_0$

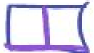
we can extend $f_t$ to all of X.

Example. 

$Y = \mathbb{R}^2$

Given 

Extend 

X×I  $M_i$

HEP same as:

Every $F: M_i \to Y$ $(i: A \hookrightarrow X)$

extends to $\tilde{F}: X \times I \to Y$

---

A retraction of a space X to a subspace A is a contin. map $r: X \to A$

s.t. $r|_A = id$.

e.g. ① $X = I, A = \bullet$ ② $X = I, A = :$ no retraction IVT

$r = $ const.

③ $X = \bullet\!-\!\bullet\!-\!\bullet$ $A = \bullet$ $X \times I$ retracts to $M_i$

Prop. (X,A) has HEP ⟺ $M_i$ is a retract of $X \times I$

Pf. (⇒) $Y = M_i$, $F = id$. ⇝ $\tilde{F}$ is retract.

(⇐) $X \times I \xrightarrow{r} M_i \xrightarrow{F} Y \rightsquigarrow \tilde{F} = F \circ r$

□

**Note.** If $X$ def ret to $A$, by $f_t$, then $f_1 : X \to A$ is a retraction.

**Lemma.** If $(X,A)$ is a CW pair then $M_i$ is a def. ret. of $X \times I$. In partic. $(X,A)$ has HEP.

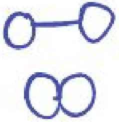
**Pf.** Special case. $X = D^n$, $A = \partial D^n$.

n=2 picture

$X \times I$ $\quad$ $M_i$: cup shape

General case: Retract each n-cell of $X^n \setminus A^n$ to $M_i : A^n \to X^n$ during $[\frac{1}{2}^{n+1}, \frac{1}{2}^n]$

Continuous, since it is cont. on each cell. □



**Prop 1.** $(X,A)$ has HEP
$A$ contractible

$\implies q : X \to X/A$ is hom. eq.

**Pf.** Need a homotopy inverse $X/A \to X$

Def Retract $A \to a$ $\quad a \in A$

Extend: $f_t : X \to X$

Since $f_1(A) = pt$, can regard $f_1$ as a map $X/A \to X$. □

For proof of second criterion, see book or 2012 notes

Office Hours

Fri 11-12, by appt.

HW p.38 5, 6 & A1 ← web site

## Fundamental Group

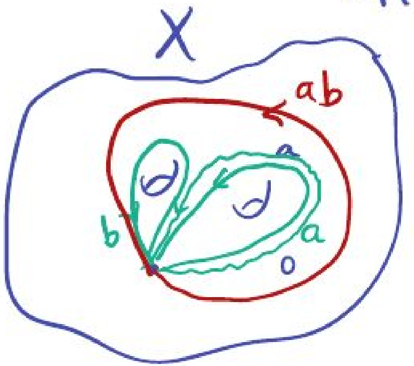
X = top. space

informally:

$\pi_1(X)$ = fund. gp of X

= group of based loops in X up to homotopy

operation: concatenation.



# Examples (without proof)

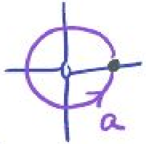
⓪ $\pi_1(\mathbb{R}^2) = 1$

① $\pi_1(S^2) = 1$.

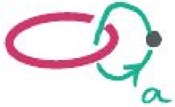
$\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$ (later)

$\Longrightarrow S^2 \neq \mathbb{RP}^2$

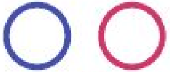
② $\pi_1(\mathbb{C} - 0) \cong \mathbb{Z}$



③ $\pi_1(\mathbb{R}^3 - \text{unknot}) \cong \mathbb{Z}$



④ $\pi_1(\mathbb{R}^3 - \text{unlink})$



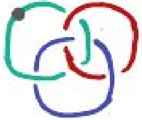
$ab = ba$?

or

$aba^{-1}b^{-1} = 1$?

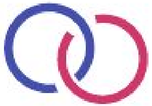
$aba^{-1}b^{-1}$

Borromean rings.



Not abelian!

⑤ $\pi_1(\mathbb{R}^3 - \text{Hopf link})$



$aba^{-1}b^{-1}$

$aba^{-1}b^{-1} = 1$

$\Longleftrightarrow ab = ba$

$aba^{-1}b^{-1}$

abelian.

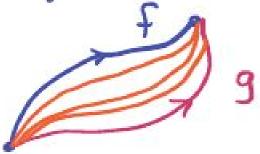
## Formal defns   $X$ = space

A path in $X$ is a map $I \to X$

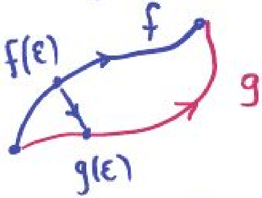
↑ always continuous.

↑ $[0,1]$

A homotopy of paths is a homotopy $f_t : I \to X$ s.t. $f_t(0)$ & $f_t(1)$ are indep. of $t$.



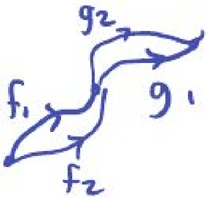
example. Any two paths $f, g$ in $\mathbb{R}^2$ with same endpts are homotopic by straight line homotopy $f_t = (1-t)f + tg$



exercise. homotopy of paths is equiv. reln.

The composition of paths $f, g$ with $f(1) = g(0)$ is the path

$$f g(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$



exercise. $f_1 \simeq f_2$, $g_1 \simeq g_2$

$\Longrightarrow f_1 g_1 \simeq f_2 g_2$

A <u>loop</u> is a path $f$ with $f(1) = f(0)$.

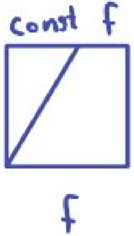
The fundamental group of $X$ based at $x_0$ is

$$\pi_1(X, x_0) = \{\text{homotopy classes of loops at } x_0\}$$

group op: composition as above.

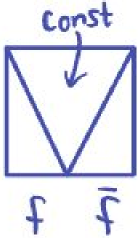
<u>Prop</u>. $\pi_1(X, x_0)$ is a group.

<u>Pf</u>. Identity: const loop.



Note: A homotopy of paths is $I \times I \to X$

Inverses:



$\bar{f}(t) = f(1-t)$

$f$ backwards.

Associativity

$(fg)h = f(gh)$

exercise: write details. □

Jan 24

# Fundamental groups

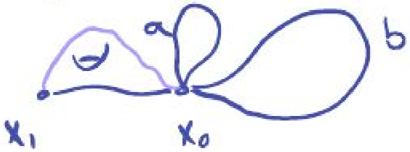
$\pi_1(X,x_0) = \{\text{loops in } X \text{ at } x_0\}/\sim$

Prop. $X$ path conn.

$x_0, x_1 \in X$

$\Rightarrow \pi_1(X,x_0) \cong \pi_1(X,x_1)$

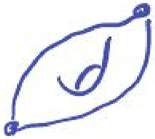
The $\cong$ given by any path $x_0$ to $x_1$



So: the $\cong$ is not canonical: different paths give diff $\cong$'s.

Say $X$ is simply connected if $\pi_1(X) = 1$

Same as: given 2 pts, any two paths between are homotopic.



Fact. $X$ contractible CW complex. $\Rightarrow$ $X$ simply conn.

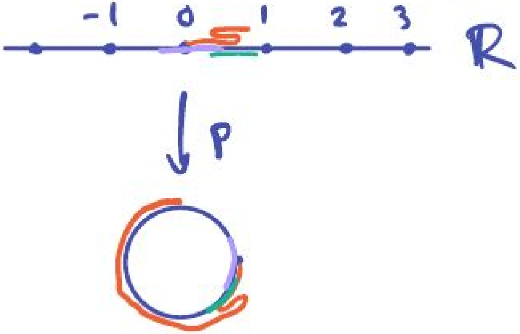
Use: $\exists$ strong def ret. to a pt when $X$ is contract. CW complex (uses HEP).

# Fundamental Group of the Circle

**Thm** $\pi_1(S^1) \cong \mathbb{Z}$

**Pf outline.** Consider

$$p: \mathbb{R} \longrightarrow S^1$$
$$t \longmapsto e^{2\pi i t}$$



Given a loop $f: I \to S^1$ want to find a lift

$$\tilde{f}: I \to \mathbb{R} \quad \text{so}$$

$p \circ \tilde{f} = f$. & $\tilde{f}(0) = 0$.

Then the map $\pi_1(S^1) \longrightarrow \mathbb{Z}$

is $\qquad [f] \longmapsto \tilde{f}(1)$.

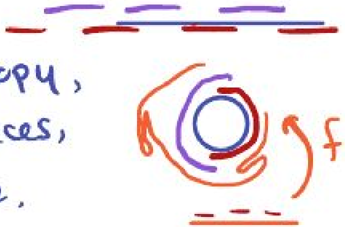
Well-def. existence/uniqueness of lifts.

Multiplicativity. easy.

Injectivity. Homotopy in $\mathbb{R}$ ~ homotopy in $S^1$ via p.

Surjectivity. easy.

Remains: lifting paths / homotopies in $S^1$ to $\mathbb{R}$.



Idea: Given a path / homotopy, cut it into small pieces, lift them one by one.

<u>Lemma</u>. Given $F: Y \times I \to S^1$

$\tilde{F}: Y \times \{0\} \to \mathbb{R}$

a lift of $F|_{Y \times \{0\}}$

$\exists!\ \tilde{F}: Y \times I \to \mathbb{R}$ lifting $F$, extending $\tilde{F}|_{Y \times \{0\}}$.

$F = p \circ \tilde{F}$

Path lifting: $Y = pt$.

Homotopy lifting: $Y = I$

<u>Proof</u> ($Y = \{y_0\}$ case)

Cover $S^1$ by $\{U_\alpha\}$ s.t. $\forall \alpha$ $p^{-1}(U_\alpha)$ is a disjoint union of open intervals, each homeo to $U_\alpha$

$F$ contin, $I$ compact $\Longrightarrow$ can choose $0 = t_0 < \cdots < t_m = 1$ s.t. $\forall i$ $F([t_i, t_{i+1}]) \subset U_\alpha$ some $\alpha$.

Induct: if $\tilde{F}$ defined on $[0, t_i]$ & $\tilde{F}(t_i) \in \tilde{U}_i$ use the homeo $h: \tilde{U}_i \to U_\alpha$

Define $\tilde{F}$ on $[t_i, t_{i+1}]$ via

$h^{-1} \circ F|_{[t_i, t_{i+1}]}$

$\square$

**Prop.** $X, Y$ path conn.

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

**Cor.** $\pi_1(T^2) = \mathbb{Z}^2$.

## Induced homomorphisms

$$\varphi : (X, x_0) \longrightarrow (Y, y_0)$$

$$\rightsquigarrow \varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

$$[f] \longmapsto [\varphi \circ f]$$

"Functoriality" :

① $(\varphi\psi)_* = \varphi_* \psi_*$

② $\mathrm{id}_* = \mathrm{id}$.

**Fact.** $\varphi$ a homeo $\Rightarrow$ $\varphi_*$ is $\cong$.

**Pf.** $\varphi_* \circ (\varphi^{-1})_* \overset{\textcircled{1}}{=} (\varphi\varphi^{-1})_* = \mathrm{id}_* \overset{\textcircled{2}}{=} \mathrm{id}$ $\square$

# APPICATIONS OF $\pi_1$

**Brouwer's Fixed Pt thm**

Every $h: D^2 \to D^2$ has a fixed pt.

Last time: $f: (X, x_0) \to (Y, y_0)$

$\rightsquigarrow f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$

**Prop.** $i: A \to X$ inclusion
$r: X \to A$ retraction

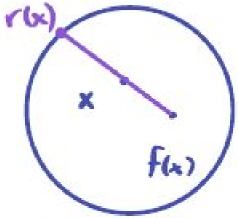
$\Longrightarrow i_*$ injective.

**Pf.** $r \circ i = \mathrm{id}$

$\Rightarrow r_* \circ i_* = \mathrm{id}$ $\square$

Jan 26

**Pf of BFPT** Suppose $h: D^2 \to D^2$ has no fixed pt. Then there is a retraction $D^2 \to S^1$



Prop $\Rightarrow$

$$\begin{array}{ccc} \pi_1(S^1) & \to & \pi_1(D^2) \\ \shortparallel & & \shortparallel \\ \mathbb{Z} & & 1 \end{array} \text{ inj.} \qquad \square$$

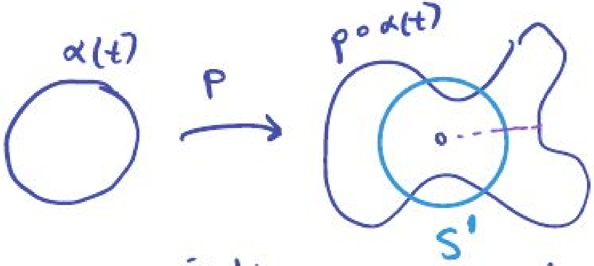
Also:
- Borsuk Ulam Thm
  any $f: S^2 \to \mathbb{R}^2$ has $f(x) = f(-x)$
- Ham sandwich thm.
- If $S^2$ = union of 3 closed sets, one has a pair of antip. pts.

# Fundamental Thm of Alg

Every nonconst. poly with coeff's in $\mathbb{C}$ has a root in $\mathbb{C}$.

**Idea.** If $\alpha(t)$ is a loop in $\mathbb{C}$ & $p(z)$ is a poly with no roots on $\alpha(t)$

$\rightsquigarrow$ loop in $S^1$ by projecting.



projection: $x \longmapsto x/\|x\|$.

For proof: modify $\alpha$ and $p$ to get a homotopy from degree $n$ loop in $S^1$ to const. loop.

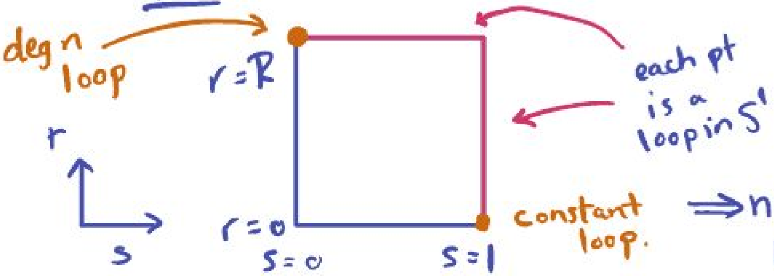
**Pf of Thm.** Say $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$
Assume no roots

Let $p_s(z) = z^n + s(a_1 z^{n-1} + \dots + a_n)$

Let $\alpha_r(t)$ = circle of radius $r$ in $\mathbb{C}$.

Let $R > |a_1| + \dots + |a_n| + 1$

**Claim.** $p_s(z)$ has no roots on $\alpha_R$



$\Longrightarrow n=0$

□

Last time:

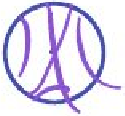
Fact. $f$ homeo $\Longrightarrow$ $f_*$ isom.

Prop. $\pi_1(S^n) = 1$, $n > 1$

Pf. $S^n - pt \underset{\text{homeo}}{\cong} \mathbb{R}^n$

(stereographic proj).

Suffices to show: any loop is homotopic to one that is not surjective.



Apply above fact about $f_*$ $\square$

Prop. $\mathbb{R}^2 \not\cong \mathbb{R}^n$ $n > 2$

Pf. $\mathbb{R}^n - \overset{\text{any}}{pt} \cong S^{n-1} \times \mathbb{R}$

(polar coords)

By Fact

$$\pi_1(\mathbb{R}^n - pt) \cong \pi_1(S^{n-1} \times \mathbb{R})$$
$$\cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$$
$$\cong \pi_1(S^{n-1})$$
$$\cong \begin{cases} \mathbb{Z} & n = 2 \\ 1 & n > 2 \end{cases}.$$

Apply Fact. and the "any" $\square$

Prop. If $\varphi : (X, x_0) \longrightarrow (Y, y_0)$ hom eq.

then $\varphi_*$ is $\cong$.

Pf. Let $\psi$ be homotopy inverse

i.e. $\psi \circ \varphi \simeq \mathrm{id}$.

isomorphism.

Remains to show: $(\psi \circ \varphi)_* =$ ~~$\mathrm{id}_* = \mathrm{id}$.~~

Or: $H_t : X \longrightarrow X$ homotopy

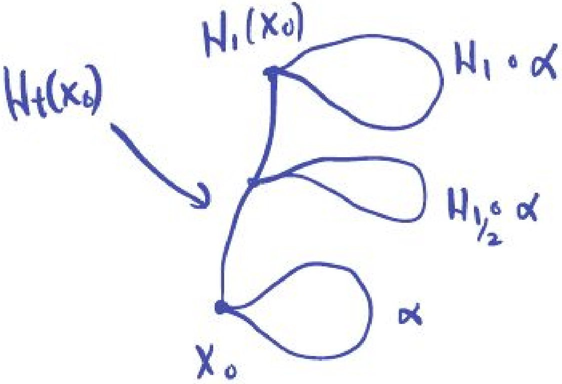
$H_0 = \mathrm{id}$

Then $(H_1)_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, H_1(x_0))$

is an isomorphism.

Proof of this last fact:

It's the change of basept isomorphism from last time.



□

Goal: compute $\pi_1$ for lots of spaces

## Free groups and free products

$F_n = \{$ freely reduced words in $x_1^{\pm 1}, \dots, x_n^{\pm 1} \}$

group op: concatenation, free reduce.

existence of $F_n$ is nontrivial! (associativity)

$G, H$ groups

$G * H$ = free product of $G$ & $H$

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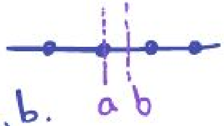
$= \{$ freely reduced words in $G, H \}$

elts look like $g_1 h_1 g_2 h_2 \cdots g_n h_n$

or $g_1 h_1 \cdots g_n$ etc.

Examples ① $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$

= symmetries of 

= alt words in a, b.

② $\mathbb{Z} * \mathbb{Z} \cong F_2$

Properties ① $G, H \leq G * H$

② $G \cap H = 1$

③ Given $G \to K$, $H \to K$ group

"univ. property"

$\exists!\ G * H \to K$

# VAN KAMPEN'S THM

$X = A \cup B$ $\quad$ $A, B$ open, path conn.

$A \cap B$ path conn.

$x_0 \in A \cap B$ basept for $X, A, B, A \cap B$

The inclusions $A, B \hookrightarrow X$ induce

$$\pi_1(A), \pi_1(B) \longrightarrow \pi_1(X)$$

By ③ on last slide we get

$$\Phi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$$

Let $N$ be normal subgp of $\pi_1(A) * \pi_1(B)$ gen by

$i_A : A \cap B \to A$

$i_B : A \cap B \to B$

$$\big((i_A)_*(w)\big)\big((i_B)_*(w)\big)^{-1} \quad \forall\, w \in \pi_1(A \cap B)$$

Then: ① $\Phi$ surjective

② $\ker \Phi = N$.

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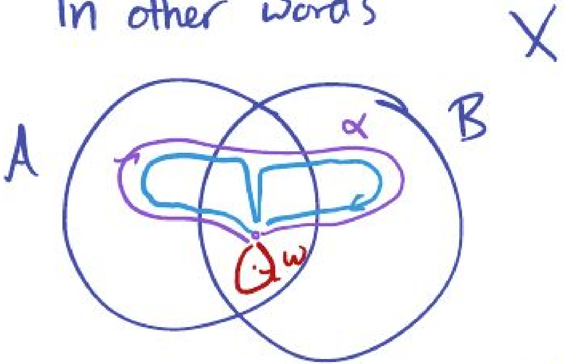
In other words



① Any $\alpha \in \pi_1(X)$ is product of loops in $A, B$.

② If $w \in \pi_1(A \cap B)$, the corresp. elts of $\pi_1(A), \pi_1(B)$ are equal.

---

Or: $\pi_1(X) = \pi_1(A) * \pi_1(B) \,/\, N$

# Examples

① $\pi_1(S^1 \vee S^1)$

$\pi_1(A) * \pi_1(B) /_N \cong \mathbb{Z} * \mathbb{Z} / 1$

$\cong F_2$

Induction: $\pi_1(\bigvee_n S^1) \cong F_n$.

$\Longrightarrow \pi_1(\mathbb{R}^2 - n \text{ pts}) \cong F_n$

$\pi_1(\mathbb{R}^3 - \text{unlink}) \cong F_n$.

$\pi_1(\text{any graph}) \cong F_n$ some $n$.

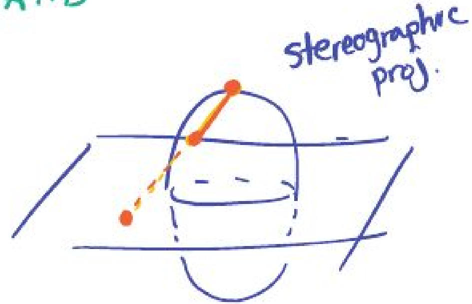
② $\pi_1(S^n) \cong 1$

$A = S^n$ - north pole

$B = S^n$ - south pole.

$A \cap B \simeq S^{n-1}$.

For $n=2$: $1 * 1 /_1 = 1$



③ $\pi_1(S^3 - (p,q) \text{ torus knot})$

$\cong \langle x, y : x^p = y^q \rangle$

read in Hatcher

## Van Kampen in terms of presentations

$\pi_1(A) = \langle S_1 \mid R_1 \rangle$

$\pi_1(B) = \langle S_2 \mid R_2 \rangle$

$\pi_1(A) * \pi_1(B) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \rangle$

Choose a gen set $S_3$ for $\pi_1(A \cap B)$.

For each $w \in S_3$ write it as a product $w_1$ of elts of $S_1$ & as a product $w_2$ of elts of $S_2$

Let $R_3$ be the set of relators $w_1 w_2^{-1}$ constructed in this way.

Then:

$$\pi_1(X) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \amalg R_3 \rangle$$

Preview of next time:

Gluing a disk to X

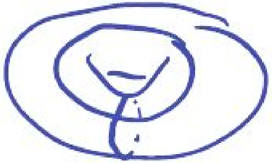
⇝ adding a relation to $\pi_1(X)$

examples ① $X = S^1$

$\pi_1(X) = \langle a \mid \rangle \cong \mathbb{Z}$

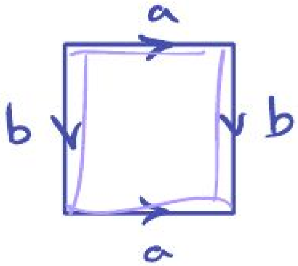
adding disk: $\pi_1(\hat{X}) = \langle a \mid a \rangle = 1$

②



$X = S^1 \vee S^1$

$\pi_1(X) \cong F_2 = \langle a, b \mid \ \rangle$



$\pi_1(\hat{X}) = \langle a, b \mid aba^{-1}b^{-1} \rangle$

$= \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2.$

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# Basics of group presentations

$G = \langle S \mid R \rangle$ means:

① $G$ is gen by $S$

② Two words in $S^{\pm 1}$ are equal in $G$ iff they differ by finitely many elts of $R$.

**Fact.** If $G = \langle S \mid R \rangle$ then

$$G \cong F(S) / \langle\langle R \rangle\rangle$$

Here: each elt of $R$ is a relator like $aba^{-1}b^{-1}$ instead of a relation $ab = ba$.

## Examples

$F_2 = \langle a, b \mid \ \rangle$

$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

$\mathbb{Z}/n = \langle a \mid a^n = 1 \rangle$

Check: modding out by $aba^{-1}b^{-1}$ is same as saying $ab = ba$

$$\begin{aligned} ba &= ba\left((ba)^{-1}(aba^{-1}b^{-1})(ba)\right) \\ &= ab \end{aligned}$$

# VAN KAMPEN's THM

$X = A \cup B$  A, B open, path conn.
$A \cap B$ path conn.

$x_0 \in A \cap B$ basept for $X, A, B, A \cap B$

The inclusions $A, B \hookrightarrow X$ induce

$$\pi_1(A), \pi_1(B) \longrightarrow \pi_1(X)$$

By ③ ~~on last slide~~ we get

$$\Phi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$$

Let $N$ be normal subgp of $\pi_1(A) * \pi_1(B)$ gen by

$$((i_A)_*(w))((i_B)_*(w))^{-1} \quad \forall\, w \in \pi_1(A \cap B)$$

$i_A : A \cap B \to A$
$i_B : A \cap B \to B$

Then: ① $\Phi$ surjective
② $\ker \Phi = N$.

Or: $\pi_1(X) = \pi_1(A) * \pi_1(B) \,/\, N$

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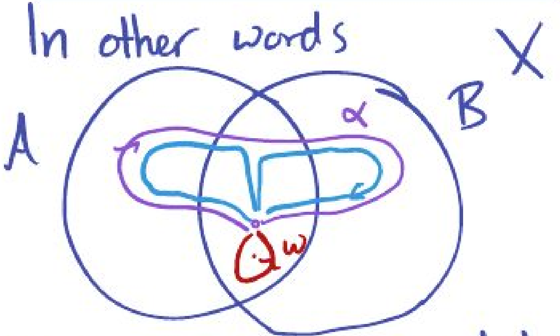
In other words



① Any $\alpha \in \pi_1(X)$ is product of loops in $A, B$.

② If $w \in \pi_1(A \cap B)$, the corresp. elts of $\pi_1(A), \pi_1(B)$ are equal.

and: all relations come from this & those in $A, B$

# Van Kampen in terms of presentations

$\pi_1(A) = \langle S_1 | R_1 \rangle$, $\pi_1(B) = \langle S_2 | R_2 \rangle$

$\Rightarrow \pi_1(A) * \pi_1(B) = \langle S_1 \amalg S_2 | R_1 \amalg R_2 \rangle$

Choose a gen set $S_3$ for $\pi_1(A \cap B)$.

For each $\omega \in S_3$ write it as a product $\omega_1$ of elts of $S_1$ & as a product $\omega_2$ of elts of $S_2$

Let $R_3$ be the set of relations $\omega_1 = \omega_2$ constructed in this way.

Then:

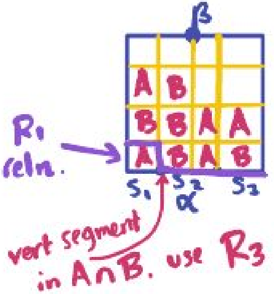
$$\pi_1(X) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \amalg R_3 \rangle$$

# Proof of VKT

① is compactness of $I$ plus above picture.

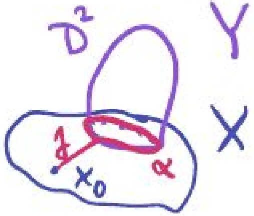
② is... Say $\alpha, \beta$ are words in $S_1 \amalg S_2$ that are equal in $\pi_1(X)$. Must show they differ by $R_1 \amalg R_2 \amalg R_3$

Since $\alpha = \beta$ in $\pi_1 X$, get homotopy.



By compactness, chop into smaller squares, each mapping to A or B. Push across one at a time. Contemplate.

# ATTACHING DISKS



X path conn. based at $x_0$

Attach 2-cell $D^2$ via

$\varphi : S^1 \to X \rightsquigarrow Y = X \cup \text{disk}.$

Choose path $\gamma$ from $x_0$ to $\varphi(S^1)$

The loop $^{\alpha}$ $\gamma\varphi(S^1)\bar{\gamma}$ is null-hom. in Y.

Let $N = \langle\langle \overset{\alpha}{\cancel{\text{this loop}}} \rangle\rangle$

**Prop.** Inclusion $X \hookrightarrow Y$ induces $\pi_1(X) \twoheadrightarrow \pi_1(Y)$ with kernel $N$.

So: $\pi_1(Y) \cong \pi_1(X)/N$.

**Proof.** Van Kampen.

$x \in D^2$. $A = Y - x$.

$B = \text{int } D^2$

$$A \cap B = \text{int } D^2 - x \simeq S^1$$

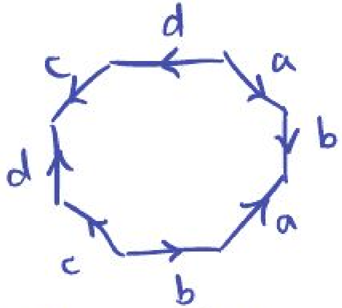
$$\text{VKT} \Rightarrow \pi_1(Y) = \pi_1(X) * \pi_1(D^2) \Big/ N$$

$$\cong \pi_1(X)/N$$

where $N = \langle\langle \alpha \cdot 1^{-1} \rangle\rangle$ □

## Examples

① $M_2 =$



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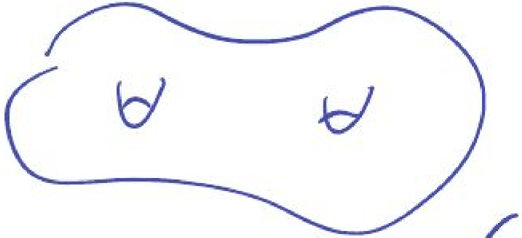


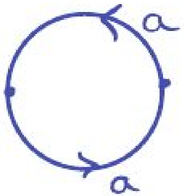
$= \bigvee_4 S^1 \cup \text{disk}$

$\rightsquigarrow \pi_1(M_2) = F_4/\langle\langle \partial D^2 \rangle\rangle = \langle a,b,c,d \mid \overset{aba^{-1}b^{-1}}{[a,b]}[c,d] = 1 \rangle$

Similar: $\pi_1(M_g) = \langle x_1, y_1, \dots, x_g, y_g \mid [x_1,y_1]\cdots[x_g,y_g] = 1 \rangle$

Conseq. $\pi_1(M_g)^{ab} = \mathbb{Z}^{2g} \Rightarrow$ If $g \neq h$ then $M_g \not\cong M_h$.

② $\mathbb{RP}^2 =$

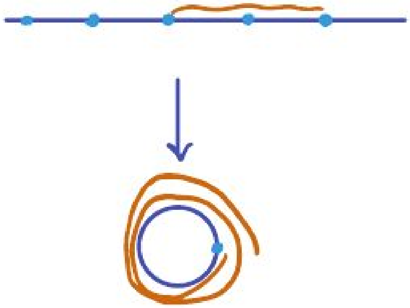


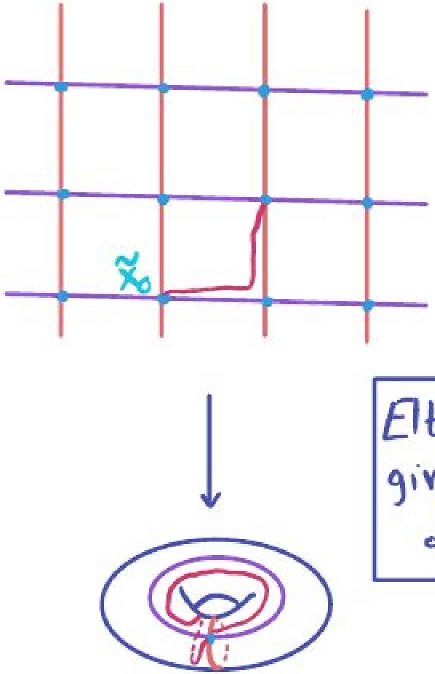
$\Rightarrow \pi_1(\mathbb{RP}^2) = F_1/\langle\langle a^2 \rangle\rangle \cong \mathbb{Z}/2$.

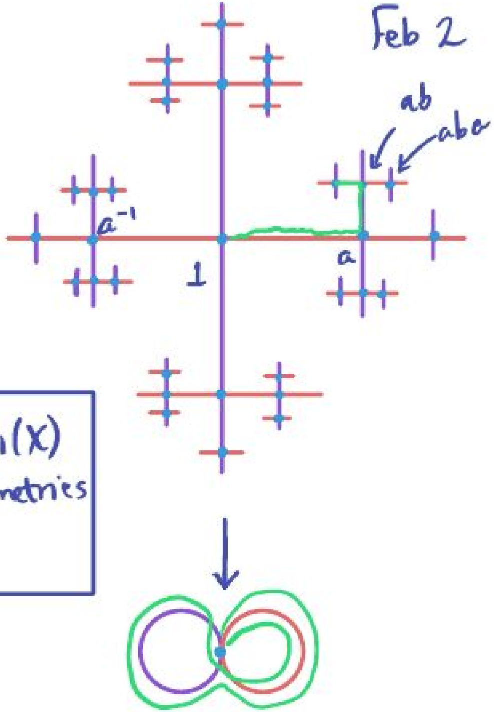
So: $\mathbb{RP}^2 \not\cong S^2$

# COVERING SPACES

Feb 2







Elts of $\pi_1(X)$ give symmetries of $\tilde{X}$

The key to proving $\pi_1(S^1) \cong \mathbb{Z}$ is path/homotopy lifting from $S^1$ to $\mathbb{R}$. We can do this for other spaces...

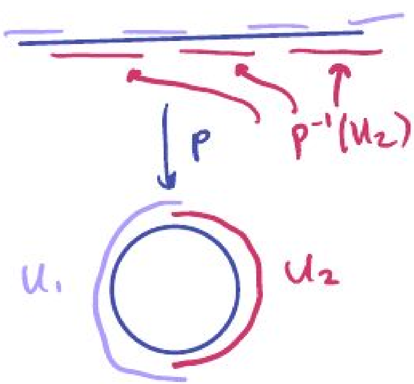
# Covering spaces

A cov sp of $X$ is connected $\tilde{X}$ with $p : \tilde{X} \to X$ satisfying:

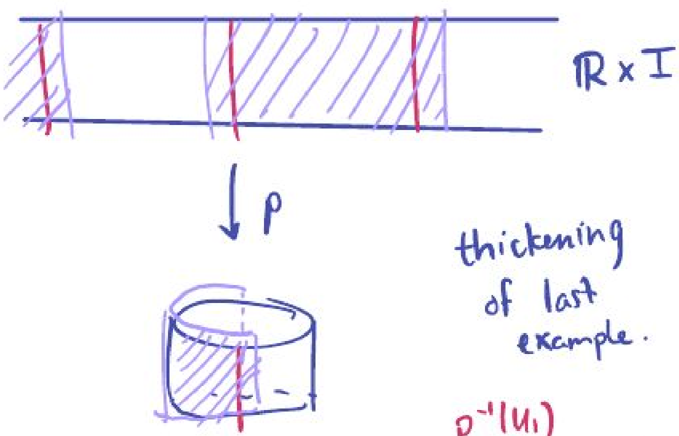
$\exists$ open cover $\{U_\alpha\}$ of $X$ s.t. each $p^{-1}(U_\alpha)$ is a disj. union of open sets, each homeo to $U_\alpha$.

## Examples

①

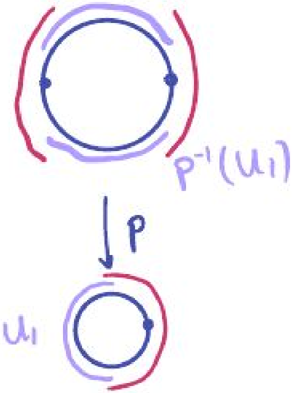


②

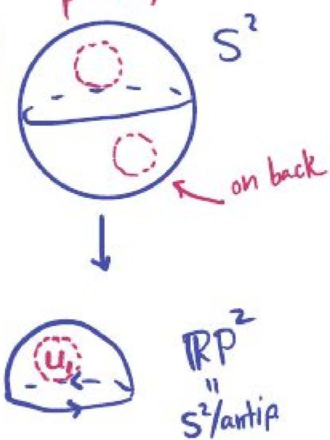


thickening of last example.

③



④



A universal covering space is one that is simply connected

examples: $\mathbb{R} \to S^1$ $\quad \mathbb{R}^2 \to T^2$

$S^2 \to \mathbb{RP}^2$ $\quad T_4 \to S^1 \vee S^1$

Will show: existence/uniqueness.

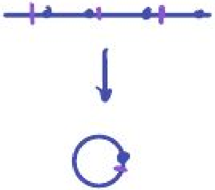
We'll also see:

① $\pi_1(X) \longleftrightarrow$ symmetries of univ cover

② subgroups of $\pi_1(X)$ $\longleftrightarrow$ covers of $X$

example: $S^1$

① via path lifting, ② via path proj.



## Fundamental Theorem

$p : \tilde{X} \to X$ covering map

$G(\tilde{X})$ = deck transformation group

= $p$-equivariant symmetries of $\tilde{X}$

i.e.

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

$p \circ f = p$

$H = p_* \pi_1(\tilde{X})$

$N(H)$ = normalizer in $\pi_1(X)$ of $H$

= largest subgp of $\pi_1(X)$ s.t. $H \trianglelefteq N(H)$

= elts of $\pi_1(X)$ that conj $H$ to itself

If $H \trianglelefteq \pi_1(X)$, $N(H) = \pi_1(X)$.

**Thm.** $1 \to H \to N(H) \to G(\tilde{X}) \to 1$

is exact. The map $N(H) \to G(\tilde{X})$ is

$\gamma \mapsto$ unique deck transf.

taking $\tilde{x}_0$ to $\tilde{\gamma}(1)$

Exact sequence: Image of each map is kernel of next one.

Short exact seq:

$$1 \to K \xrightarrow{i} G \xrightarrow{f} Q \to 1$$

exactness same as: $i$ inj.

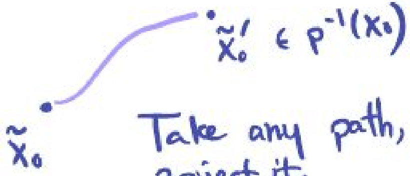
$f$ surj.

same as: $Q \cong G/K$.

**Cor.** $H = 1 \Rightarrow \pi_1(X) \cong G(\tilde{X})$

**Cor.** $H \trianglelefteq \pi_1(X) \Leftrightarrow G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$

Pf. ($\Rightarrow$)



Take any path, project it.

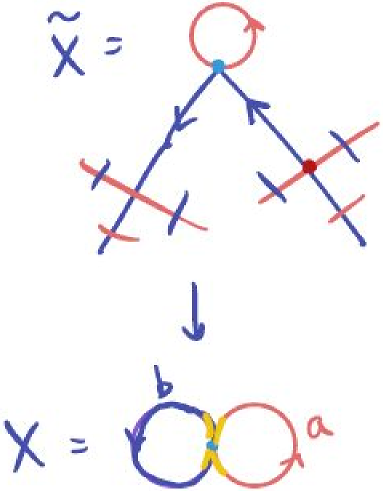
$\rightsquigarrow$ elt of $\pi_1(X)$

Also: There is a bijection:

$$\left\{\begin{matrix}\text{based cov sp} \\ \text{of } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{subgps} \\ \text{of } \pi_1(X)\end{matrix}\right\}$$

top. gp thy.

# Example



$\pi_1(\tilde{X}) = \mathbb{Z}$

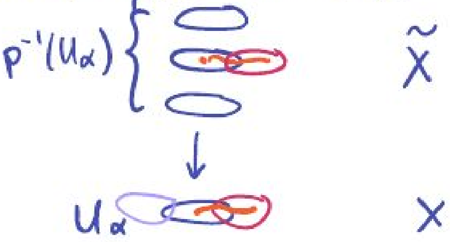
$H = \mathbb{Z} = \langle a \rangle \ntrianglelefteq F_2.$

# Covering spaces

Feb 4

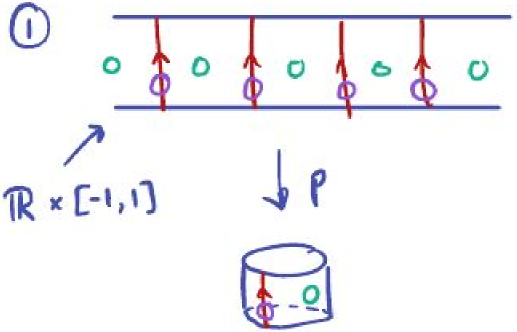
A covering space of connected $X$ is a connected space $\tilde{X}$ with a map $p : \tilde{X} \longrightarrow X$ satisfying:

$\exists$ open cover $\{U_\alpha\}$ of $X$ s.t. each $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to $U_\alpha$.



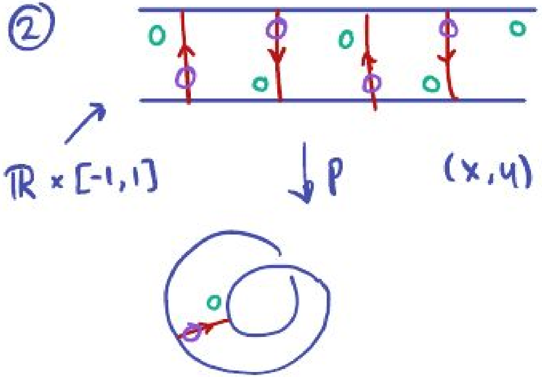
## Examples

①



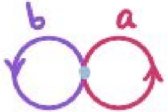
$(x,y) \sim (x+n, y)$

②



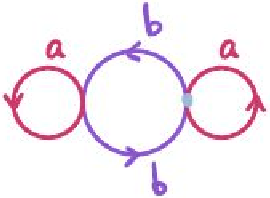
$(x,y) \sim (x+n, -y)$

# Examples

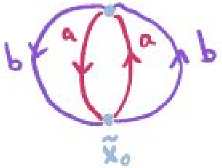
$X =$ 

$\tilde{X}$

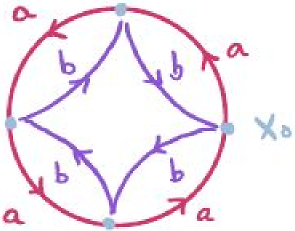
$p_*(\pi_1(\tilde{X})) \leq \pi_1(X)$



$\langle a, b^2, bab^{-1} \rangle$
$= \langle a, b^2, bab \rangle$



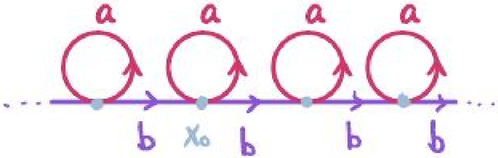
$\langle ba, b^2, ba^{-1} \rangle$



you.

$\tilde{X}$

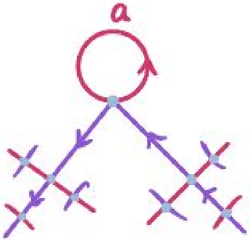
$p_*(\pi_1(\tilde{X}))$



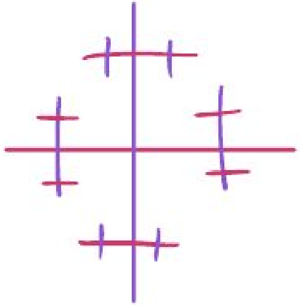
$\langle b^k a b^{-k} \rangle$

Will show $p_*$ inj.

$\Rightarrow F_\infty \leq F_2$



$\langle a \rangle$



$1$

# Lifting Properties

$p : \tilde{X} \to X$ cov sp.

A lift of $f : Y \to X$ is an $\tilde{f} : Y \to \tilde{X}$ s.t. $p \circ \tilde{f} = f$ (like lifting paths from $S^1$ to $\mathbb{R}$).

**Prop 1.** (Homotopy lifting property) Given a homotopy $f_t : Y \to X$ and $\tilde{f}_0 : Y \to \tilde{X}$ lifting $f_0$ $\exists!$ $\tilde{f}_t$ lifting $f_t$.

Special cases ① $Y = pt. \rightsquigarrow f_t =$ path

② $Y =$ interval $\rightsquigarrow f_t =$ homotopy of paths

**Pf.** Same as $S^1$.

**Cor.** $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ injective.

**Pf.** If $p_*(\alpha) = 1 \in \pi_1(X) \rightsquigarrow$ homotopy in $X$ to const. Lift the homotopy $\Rightarrow \alpha = 1$.

**Note.** $p_*(\pi_1(\tilde{X}))$ is exactly the subgp of $\pi_1(X)$ consisting of loops that lift to loops.

## Degree of a cover

$|p^{-1}(x)|$ is locally const (as a fn of $x \in X$) hence const.

This "number" is the degree of $p$.

**Cor.** $X, \tilde{X}$ path conn.

$$\deg p = [\pi_1(X) : p_*\pi_1(\tilde{X})]$$

topology group theory.

**Pf.** Let $H = p_*\pi_1(\tilde{X})$

deg p of these

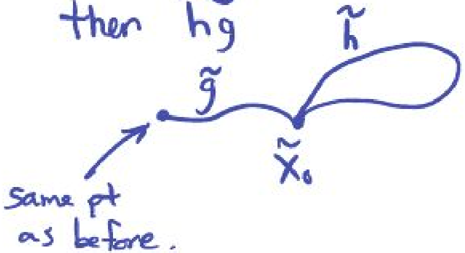
Define $\{\text{right cosets of } H\} \longrightarrow p^{-1}(x_0)$

$$Hg \longmapsto \tilde{g}(1)$$

Key pt: well def.

If $hg$ is another rep of $Hg$

then $\widetilde{hg}$



In first graph example, coset reps are:

$$1, b$$

So: $\underset{F_3}{\underset{\|}{\langle a, b^2, bab^{-1}\rangle}} \le F_2$

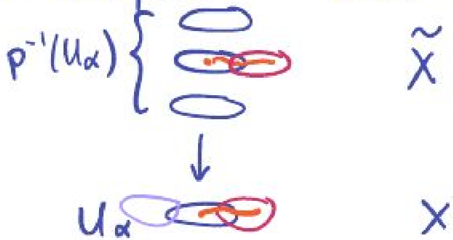
Index is 2, $b$ is a coset rep.

# Covering spaces

connected

A covering space of $X$ is a connected space $\tilde{X}$ with a map $p : \tilde{X} \to X$ satisfying:

$\exists$ open cover $\{U_\alpha\}$ of $X$ s.t. each $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to $U_\alpha$.



# Lifting Properties

Feb 7

A lift of $f : Y \to X$ is an $\tilde{f} : Y \to \tilde{X}$ s.t. $p \circ \tilde{f} = f$

**Prop 1.** (Homotopy lifting property) Given a homotopy $f_t : Y \to X$ and $\tilde{f}_0 : Y \to \tilde{X}$ lifting $f_0$ $\exists!$ $\tilde{f}_t$ lifting $f_t$.

Special cases: paths, homotopy of paths

**Cor.** $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ injective.

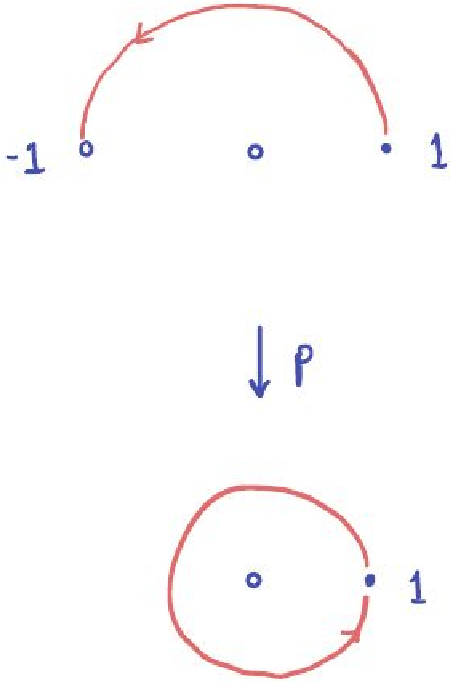
**Cor.** $X, \tilde{X}$ path conn.

$$\deg p = [\pi_1(X) : p_* \pi_1(\tilde{X})]$$

# An important covering space

$$\begin{aligned} p: \mathbb{C}\setminus 0 &\longrightarrow \mathbb{C}\setminus 0 \\ z &\longmapsto z^2 \end{aligned}$$

Non-uniqueness of square roots corresponds to the fact that this is a nontrivial cover.



$\pi_1(\tilde{X}) = \mathbb{Z}$

$\pi_1(X) = \mathbb{Z}$

$p_* : \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

$p_*(\mathbb{Z}) = 2\mathbb{Z}$

**Prop 2.** (Lifting criterion) $Y$ connected, locally path conn. We can lift $f : (Y, y_0) \to (X, x_0)$ to $\tilde f : (Y, y_0) \to (\tilde X, \tilde x_0)$ iff

$$f_*(\pi_1(Y)) \leq p_* \pi_1(\tilde X).$$

$$\begin{tikzcd} & \tilde X \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde f"] \arrow[r, "f"'] & X \end{tikzcd}$$

Already know we can lift when $Y = I, I \times I$ (special case).

Pf. (⇒) $\tilde f$ exists $\Rightarrow p \circ \tilde f = f$

$\Rightarrow p_* \circ \tilde f_* = f_*$ ✓

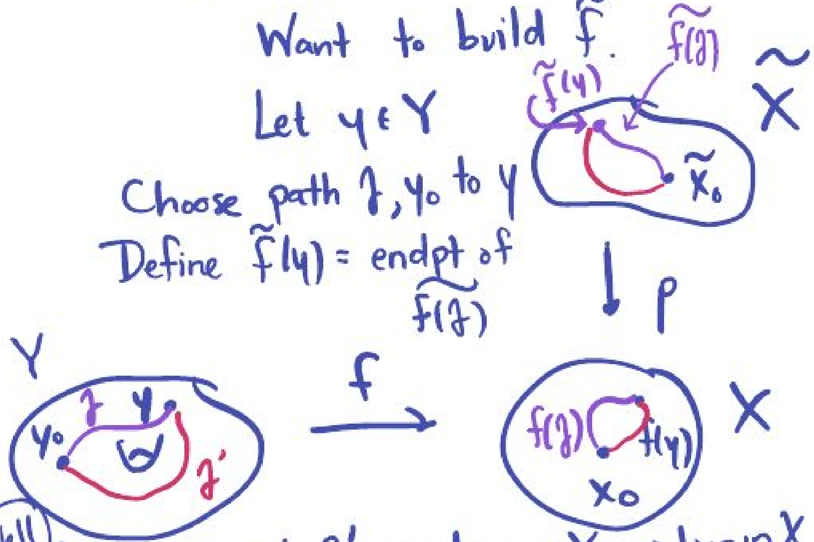
(⇐) Suppose $\operatorname{Im} f_* \subseteq \operatorname{Im} p_*$

Want to build $\tilde f$.

Let $y \in Y$

Choose path $\gamma$, $y_0$ to $y$

Define $\tilde f(y)$ = endpt of $\widetilde{f(\gamma)}$



Well def Another path $\gamma'$ ⇝ loop in $Y$ ⇝ loop in $X$ ⇝ loop in $\tilde X$ (by hypothesis!) □

**Prop 3** (Uniqueness of lifts)

Let $f : Y \to X$, $Y$ conn.

If lifts $\tilde{f}_1, \tilde{f}_2$ agree at one pt, then they are equal.

Already know the case $Y = I, I \times I$

**Pf.** Will show

$$A = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$$

is open & closed in $Y$.

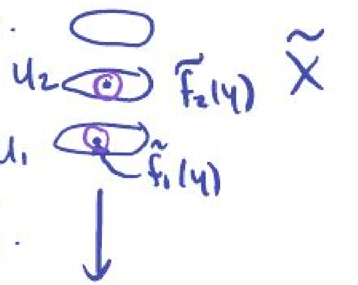
$(\Rightarrow A = Y)$.

note $A \neq \emptyset$ by assumption.

Let $y \in Y$

$U$ = open nbd of $f(y)$ as in defn of cov space.

Let $\tilde{U}_1, \tilde{U}_2$ components of $p^{-1}(U)$ containing $\tilde{f}_1(y), \tilde{f}_2(y)$.



- $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \tilde{U}_1 \neq \tilde{U}_2$ $\Rightarrow$ open nbd of $y$ not in $A \Rightarrow A$ closed.
- Similarly, $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{U}_1 = \tilde{U}_2$ $\Rightarrow A$ open. $\square$

# Classification of Covering Spaces

$$\left\{\text{based covers of } X\right\} \longleftrightarrow \left\{\begin{matrix}\text{subgps of}\\ \pi_1(X)\end{matrix}\right\}$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Want a map the other way.

First case: trivial subgp.

**Thm.** $X$ = CW complex (or any path conn, locally path conn, semilocally simply conn)

Then $X$ has a universal cover.

Pf. We construct $\tilde{X}$ directly.

Define

$$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$$

$$p : \tilde{X} \longrightarrow X$$
$$[\gamma] \longmapsto \gamma(1)$$

Topology on $\tilde{X}$:

$$\mathcal{U} = \{U \subseteq X : U \text{ path conn, open, } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

For $U \in \mathcal{U}$, $\gamma$ with $\gamma(1) \in U$

set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \eta(0) = \gamma(1)\}$

Open nbd of $[\gamma]$ in $\tilde{X}$.

Pf. We construct $\tilde{X}$ directly.

Define

$$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$$

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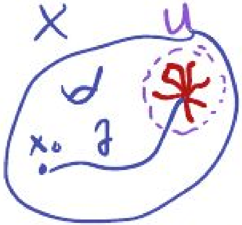

Topology on $\tilde{X}$:

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For $U \in \mathcal{U}$, $\gamma$ with $\gamma(1) \in U$

set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \eta(0) = \gamma(1)\}$

Open nbd of $[\gamma]$ in $\tilde{X}$.

Check properties of cov sp.

- continuity
- path connectivity.
- If $[\gamma'] \in U_{[\gamma]}$ then

$$U_{[\gamma']} = U_{[\gamma]}$$

Thus, for fixed $U \in \mathcal{U}$ the $U_{[\gamma]}$ partition $p^{-1}(U)$

& $p : U_{[\gamma]} \longrightarrow U$ is a homeo.

Next time: $\tilde{X}$ simply connected.

# Classification of Covering Spaces

$$\{\text{based covers of } X\} \longleftrightarrow \left\{\begin{matrix}\text{subgps of}\\ \pi_1(X)\end{matrix}\right\}$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Want a map the other way.

First case: trivial subgp.

Example

$p: \mathbb{R} \to S^1$

**Thm**. $X$ = CW complex (or any path conn, locally path conn, semilocally simply conn)

Then $X$ has a universal cover.

RECORD

Feb 9

Pf. We construct $\tilde{X}$ directly.

Define

$$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$$

$$p : \tilde{X} \longrightarrow X$$
$$[\gamma] \longmapsto \gamma(1)$$

Topology on $\tilde{X}$:

$$\mathcal{U} = \{U \subseteq X : U \text{ path conn, open}, \ \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

For $U \in \mathcal{U}$, $\gamma$ with $\gamma(1) \in U$

set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \ \eta(0) = \gamma(1)\}$

Open nbd of $[\gamma]$ in $\tilde{X}$.

$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$

$p : \tilde{X} \longrightarrow X$

$[\gamma] \longmapsto \gamma(1)$

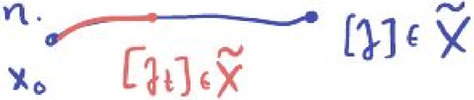
Topology on $\tilde{X}$:

$\mathcal{U} = \{U \subseteq X : U \text{ path conn, open, } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$

For $U \in \mathcal{U}$, $\gamma$ with $\gamma(1) \in U$

set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \eta(0) = \gamma(1)\}$

- $p$ is continuous
  $p^{-1}(U)$ is a union of $U_{[\gamma]}$
- $\tilde{X}$ is path conn.



- Covering space

If $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$

$\Longrightarrow$ for fixed $U$, $\{U_{[\gamma]}\}$ partition $p^{-1}(U)$

[and. $p : U_{[\gamma]} \longrightarrow U$ is a homeo b/c it induces a bijection of open sets

$V_{[\gamma]} \subseteq U_{[\gamma]} \longleftrightarrow V \subseteq U$ ]

- Simply connected

$p_*$ inj so enough: $p_* \pi_1(\tilde{X}) = 1$

$\gamma \in \operatorname{Im} p_* \Rightarrow \gamma$ lifts to loop: $\{[\gamma_t]\}$

loop $\Rightarrow [\gamma] = [\gamma_1] \sim [\gamma_0] = \text{const.}$

$\Longrightarrow \gamma = 1$ □

**Thm.** $\forall\ H \leq \pi_1(X)\ \exists$ (based) cov sp $p: \tilde{X}_H \longrightarrow X$

s.t. $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

**Pf.** Define $\tilde{X}_H$ as $\tilde{X}/\sim$ :

$[\gamma] \sim [\gamma']$ if $\gamma\overline{\gamma'} \in H$

exercise: this is equiv reln.

Check: this is a covering space.

Check: $p_* \pi_1(\tilde{X}_H, \tilde{x}_0) = H$. $\tilde{x}_0 = [\text{const loop}]$

$\gamma \in \operatorname{Im} p_* \Longleftrightarrow \{[\gamma_t]\}$ a loop in $\tilde{X}_H$

$\Longleftrightarrow \underset{\overset{\shortparallel}{\text{id}}}{\text{const}} = [\gamma_0] \sim [\gamma_1] = \gamma$

$\Longleftrightarrow \gamma \in H$ $\square$

To finish classification, need uniqueness of $\tilde{X}_H$.

**Cor.** All subgps of free gps are free.

**Pf.** $F = \pi_1(\Gamma)$ $\Gamma$ = graph = $\vee S^1$

Any subgp of $F$ is $\pi_1$ of a cover of $\Gamma$, which is a graph $\square$

**Def.** Cov sp's $\tilde{X}_1, \tilde{X}_2$ are isomorphic if there is a homeo $f: \tilde{X}_1 \to \tilde{X}_2$ with $p_1 = p_2 f$ :

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & \circlearrowright & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$$\left( \begin{tikzcd} & \tilde{X}_2 \arrow[d, "p_2"] \\ Y \arrow[ur, dashed, "\tilde{p}_1"] \arrow[r, "p_1"'] & X \end{tikzcd} \right)$$

(i.e. $f$ preserves "fibers")

↑ preimage of pt.

**Example**

$X =$ 

$\mathrm{Im}(p_1)_*$
$\shortparallel$
$\mathbb{Z} \times 1$

$\tilde{X}_1 =$  $S^1 \times \mathbb{R}$

$\tilde{X}_2 = \mathbb{C}^*$

**Prop.** $\tilde{X}_1 \cong \tilde{X}_2 \iff \mathrm{Im}(p_1)_* = \mathrm{Im}(p_2)_*$.

**Pf.** $(\Rightarrow)$ easy. (use: $f_*$ is $\cong$).

$(\Leftarrow)$ Lifting criterion $\rightsquigarrow$

lift $p_1$ to $\tilde{p}_1$ : $p_2 \tilde{p}_1 = p_1$

by symmetry: $p_1 \tilde{p}_2 = p_2$

Note: $\tilde{p}_1 \tilde{p}_2$ is a lift of $p_2$

$p_2 \tilde{p}_1 \tilde{p}_2 = p_1 \tilde{p}_2 = p_2$

Uniqueness of lifting + $\tilde{p}_1 \tilde{p}_2(\tilde{x}_2) = \tilde{x}_2$

$\Longrightarrow \tilde{p}_1 \tilde{p}_2 = id$

$\Longrightarrow$ both homeos. $\square$

# THE FUNDAMENTAL THM

Fix $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$

$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

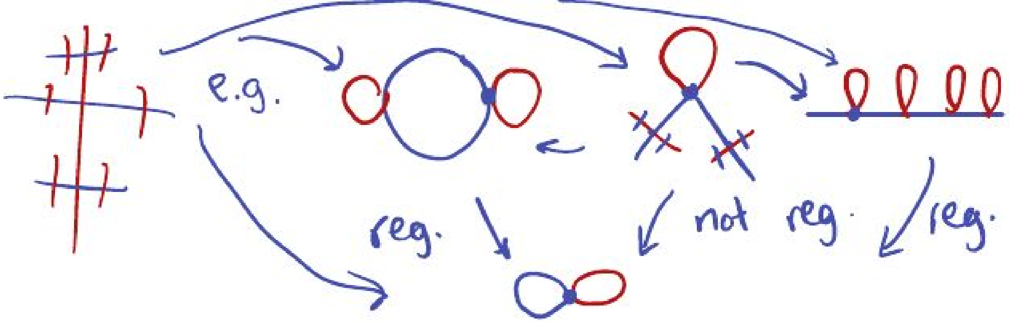
$N(H)$ = normalizer of $H$

$= \{g : gHg^{-1} = H\}$

$G(\tilde{X})$ = gp of deck transf.

= gp of isomorphisms of $\tilde{X} \circlearrowleft$

Say $p$ is regular if $G(\tilde{X})$ acts transit. on $p^{-1}(x_0)$.



**Prop.** $\tilde{X}$ reg $\iff$ $H$ normal.

**Thm.** $G(\tilde{X}) \cong N(H)/H$

i.e.

$$1 \longrightarrow H \longrightarrow N(H) \longrightarrow G(\tilde{X}) \longrightarrow 1$$

Feb 11

Last time we proved:

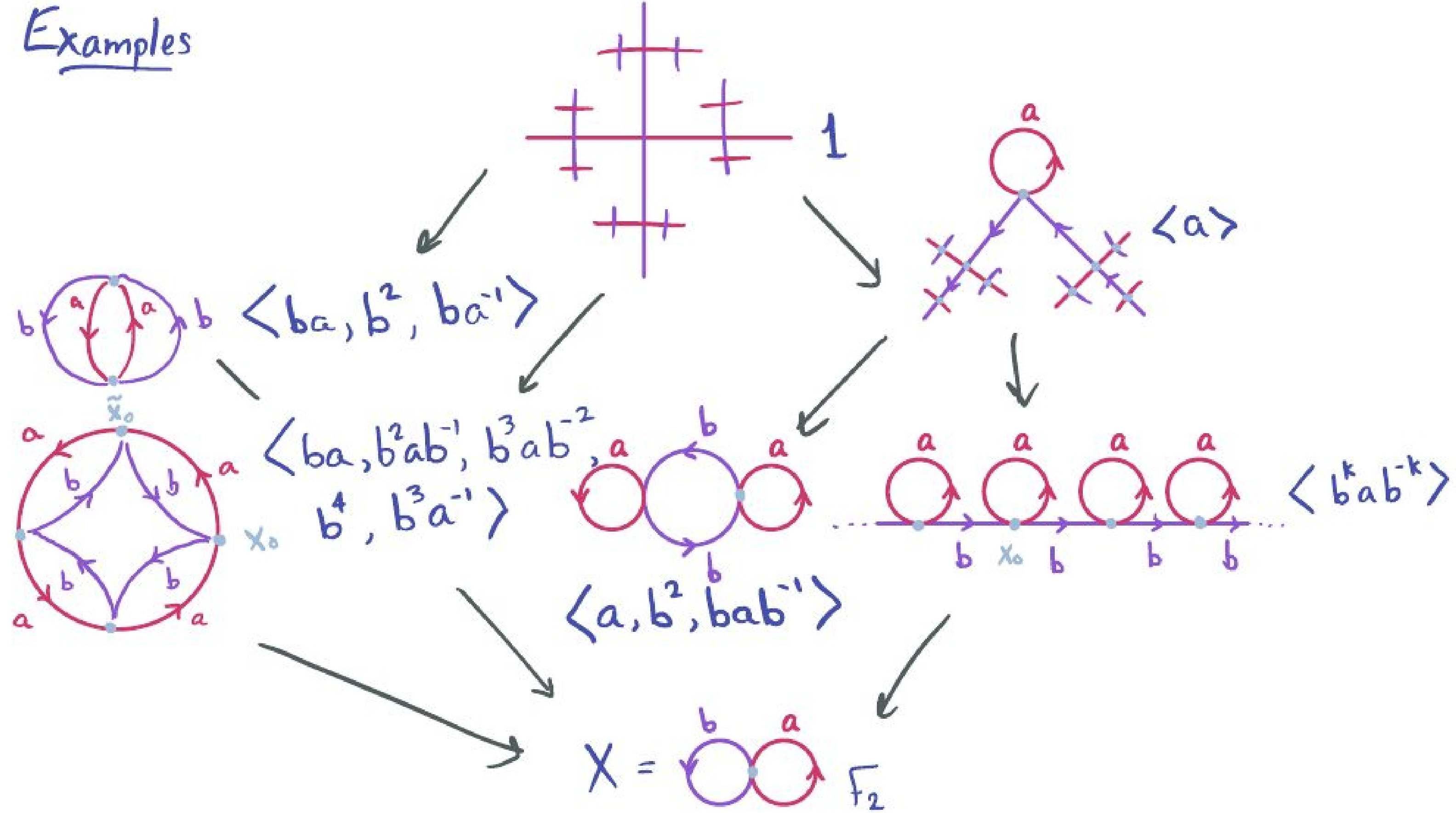
$$\left\{\begin{matrix}\text{based cov sp's}\\ \text{of } X\end{matrix}\right\}\Big/\sim \;\longleftrightarrow\; \left\{\begin{matrix}\text{subgps of}\\ \pi_1(X)\end{matrix}\right\}$$

It follows that the poset of subgps corresponds to the poset of covers:

$$\begin{tikzcd}
 & \pi_1(X) & & & & \tilde{X} \arrow[dl] \arrow[dr] & \\
A \arrow[ur, hook] & & B \arrow[ul, hook] & \Longrightarrow & \tilde{X}_C \arrow[d] \arrow[drr] & & \tilde{X}_D \arrow[d] \\
C \arrow[u, hook] \arrow[urr, hook] & & D \arrow[u, hook] & & \tilde{X}_A \arrow[dr] & & \tilde{X}_B \arrow[dl] \\
 & 1 \arrow[ul, hook] \arrow[ur, hook] & & & & X &
\end{tikzcd}$$

"contravariant functor"

# Examples


1
$\langle a \rangle$
a
$\langle ba, b^2, ba^{-1} \rangle$
$\langle ba, b^2ab^{-1}, b^3ab^{-2}, b^4, b^3a^{-1} \rangle$
$\tilde{x}_0$
$x_0$
$\langle a, b^2, bab^{-1} \rangle$
$\langle b^kab^{-k} \rangle$
$X =$
$F_2$
a
b

# Isomorphisms

Cov sp's $\tilde{X}_1, \tilde{X}_2$ are isomorphic if there is:

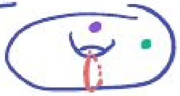
$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X}_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

preimage of pt.

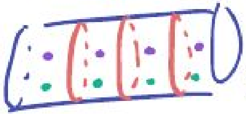
(i.e. f preserves "fibers")

$$\begin{tikzcd}
& \tilde{X} \arrow[d] \\
Y \arrow[ur, dashed] \arrow[r] & X
\end{tikzcd}$$

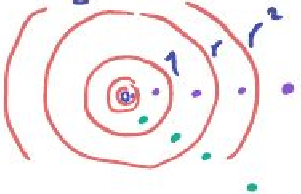
Example

$X =$ 

$\mathrm{Im}(p_1)_*$
$\shortparallel$
$\mathbb{Z} \times 1$

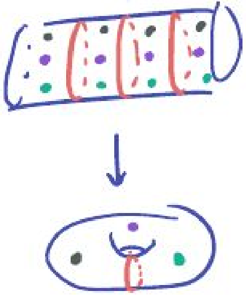
$\tilde{x}_1 =$  $S^1 \times \mathbb{R}$

$\tilde{X}_2 = \mathbb{C}^*$



## Deck Transformations

A deck transformation of a cover is an automorphism (self-isomorphism). $G(\tilde{X}) = \{\text{deck t's}\}$

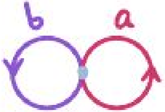


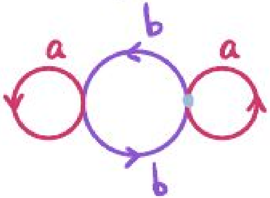
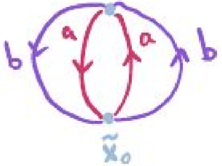
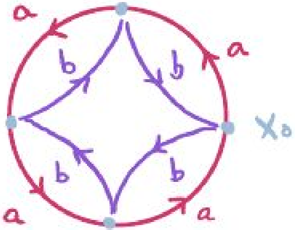
What are all the deck transf's?

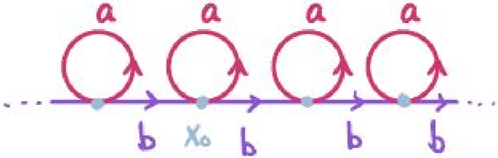
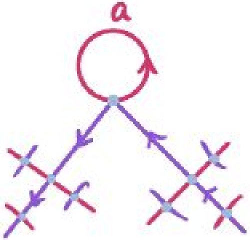
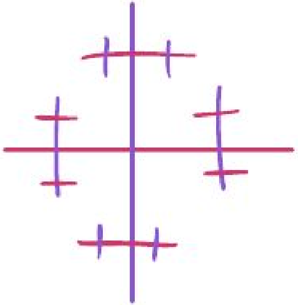
Translation by $\mathbb{Z}$

Uniqueness of lifts $\Longrightarrow$ Deck T's determined by one pt.

# Examples

$X =$ 

| $\tilde{X}$ | $G(\tilde{X})$ | |
|---|---|---|
|  | $\mathbb{Z}/2$ | rotate by $\pi$. |
|  | $\mathbb{Z}/2$ | rotate by $\pi$ |
|  | $\mathbb{Z}/4$ | rotate by $\pi/2$ |

| $\tilde{X}$ | $G(\tilde{X})$ |
|---|---|
|  | $\mathbb{Z}$ |
|  | 1 |
|  | $F_2$ |

# THE FUNDAMENTAL THM

Fix $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$

$$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong \pi_1(\tilde{X})$$

$N(H)$ = normalizer of $H$

$$= \{\gamma : \gamma H \gamma^{-1} = H\} \subseteq \pi_1(X).$$

There is a map:

$$F : N(H) \longrightarrow G(\tilde{X})$$

$$\gamma \longmapsto \text{deck transf taking } \tilde{x}_0 \text{ to } \tilde{\gamma}(1).$$

Thm. $F$ is well-def & surjective, and $\ker F = H$. In other words:

$$1 \longrightarrow H \hookrightarrow N(H) \overset{F}{\twoheadrightarrow} G(\tilde{X}) \longrightarrow 1$$

is exact. In particular:

$$N(H)/_{H} \cong G(\tilde{X}).$$

Cor. If $H \trianglelefteq \pi_1(X)$ then

$$\pi_1(X)/_{\pi_1(\tilde{X})} \cong G(\tilde{X}).$$

**Key pt**: $F$ is well def.

Regard $\tilde{x}_0$ as $[\text{const}]$.

Then $p^{-1}(x_0) = \{[\gamma] : \gamma \text{ a loop}\}/\sim$

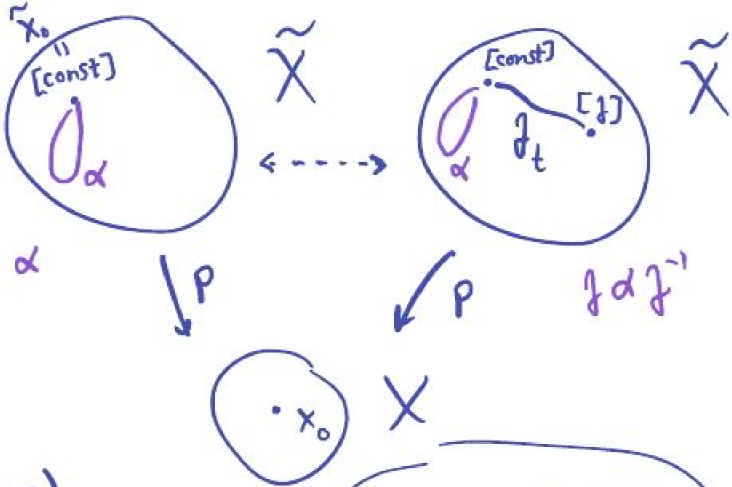
By lifting criterion:

$\exists$ deck tr $\tau$ taking $[\text{const}]$ to $[\gamma]$

$\iff p_* \pi_1(\tilde{X}, [\text{const}]) \overset{=}{\cancel{\subseteq}} p_* \pi_1(\tilde{X}, [\gamma])$.

$\iff \gamma\, p_* \pi_1(\tilde{X}, [\text{const}])\, \gamma^{-1} = p_* \pi_1(\tilde{X}, [\text{const}])$

$\iff \gamma \in N(H)$.



**Kernel** If $[\gamma] = [\text{const}]$ then deck transf is trivial.

This argument also shows surjectivity: Given $\tau \in G(\tilde{X})$. Let $\tilde{\gamma}$ be path $\tilde{x}_0$ to $\tau \cdot x_0$
Let $\gamma = p \circ \tilde{\gamma}$.

## Examples

$$F_2 / \langle a, bab^{-1}, b^2 \rangle \cong \mathbb{Z}/2$$

$$F_2 / \langle b^k a b^{-k} \rangle \cong \mathbb{Z}$$

**Regular cov sp:** $G(\tilde{X})$ acts trans. on $p^{-1}(x_0)$.

**Prop.** $\tilde{X}$ regular

$\iff H$ normal

$\left( \iff N(H) = \pi_1(X) \right)$

# Covering spaces via actions

An action of a gp $G$ on a space $Y$ is:

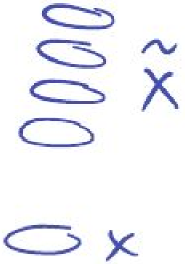
$$G \longrightarrow \text{Homeo}(Y)$$

This is a cov sp action if

$\forall\, y \in Y\ \exists$ nbd $U$ st.

$\{g(U)\}$ all distinct, disjoint.

Fact. The action of $G(\tilde{X})$ on $\tilde{X}$ is a cov sp action



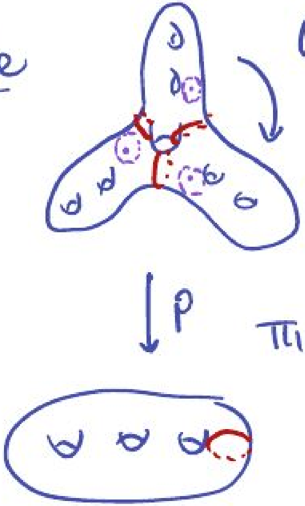
Prop. $Y$ = conn CW complex

$G \curvearrowright Y$ cov sp action

Then ① $p: Y \to Y/G$ reg cov sp.

② $G \cong G(Y)$.

Example



$G \cong \mathbb{Z}/3$

$\pi_1(S_7) \subseteq \pi_1(S_3)$

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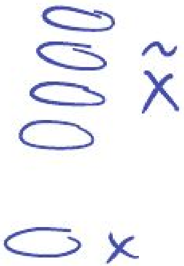

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Prop. $Y$ = conn CW complex
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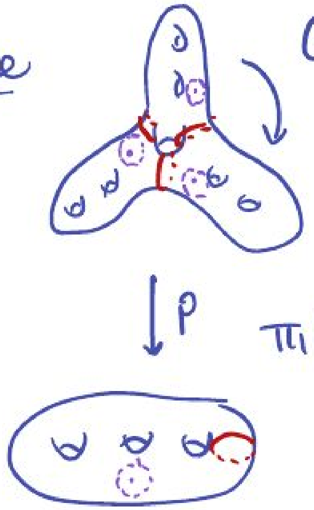
Example



$G \cong \mathbb{Z}/3$

$\pi_1(S_7) \leq \pi_1(S_3)$

**Prop.** $Y$ = conn CW complex

$G \circlearrowright Y$ cov sp action

Then ① $p : Y \to Y/G$ reg cov sp.

② $G \cong G(Y)$.

In particular:

$$G \cong \pi_1(Y/G)/p_* \pi_1(Y)$$

Further: $Y$ simply connected

$$G \cong \pi_1(Y/G).$$

**Examples of cov sp actions**

① $\mathbb{Z}/2 \circlearrowright S^2$

$\Longrightarrow \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$

Actually works for $\mathbb{Z}/2 \circlearrowright S^n$

$\Longrightarrow \pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$

② $\mathbb{Z} \circlearrowright \mathbb{R}$

$\Rightarrow \pi_1(S^1) = \mathbb{Z}$

Actually: $\mathbb{Z}^n \circlearrowright \mathbb{R}^n$

$\rightsquigarrow \pi_1(T^n) \cong \mathbb{Z}^n$

# $K(G,1)$ Spaces

Goal: groups $\longleftrightarrow$ spaces/$\sim$

homoms $\longleftrightarrow$ maps

**Def.** A $K(G,1)$ is a space $X$

with ① $\pi_1(X) \cong G$

② $\tilde{X}$ contractible.

(contractible universal cover)

**Examples** $S^1$, $T^n$, $S^1 \vee S^1$

$$\overset{\shortparallel}{K(\mathbb{Z},1)} \quad \overset{\shortparallel}{K(\mathbb{Z}^n,1)} \quad \overset{\shortparallel}{K(F_2,1)}$$

Note: $\mathbb{R}P^2$ is not a $K(\mathbb{Z}/2,1)$.

What about $\mathbb{Z}/m\mathbb{Z}$?

$$\mathbb{Z}/m\mathbb{Z} \circlearrowright S^\infty \subseteq \mathbb{C}^\infty$$

$$(z_i) \longmapsto e^{2\pi i/m}(z_i)$$

This is a cov sp. action.

$\Longrightarrow$ quotient is $K(\mathbb{Z}/m,1)$,

since $S^\infty \simeq *$.

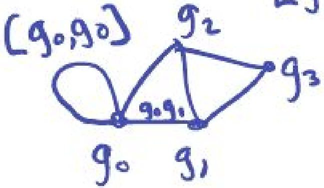
Later: Any $K(\mathbb{Z}/m,1)$ is $\infty$-dim'l!

# Construction of $K(G,1)$ spaces

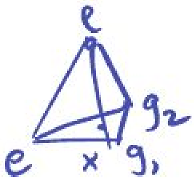
**Prop.** Every group $G$ has a $K(G,1)$.

**Pf.** Define $\Delta$-complex $EG$ with

n-simplices $\longleftrightarrow$ ordered $(n+1)$-tuples $[g_0,\dots,g_n]$ $g_i \in G$.



**Claim.** $EG$ is contractible.



**Pf.** Slide each $x \in [g_0,\dots,g_n]$ along line segment in $[e,g_0,\dots,g_n]$ to $e$.

Not a def ret b/c. $[e]$ moves along $[e,e]$

Now: $G \circlearrowright EG$ by left mult.

Claim: This is a cov sp. action

Pf: you!

So: $BG = EG/G$ is a $K(G,1)$. $\square$

This gives an $\infty$-dim $K(G,1)$.

Often: goal is to find the <u>right</u> $K(G,1)$.

# Homomorphisms as maps

**Prop.** $X$ = connected CW complex maybe a $K(H,1)$.

$Y = K(G,1)$

Every homomorphism $\pi_1(X,x_0) \longrightarrow G$ is induced by a map $X \to Y$. The map is unique up to homotopy.

**Examples** ① $\pi_1(M_g) \longrightarrow \mathbb{Z}^{2g}$ $g \geq 1$ abelianization.

$\rightsquigarrow M_g \longrightarrow T^{2g}$ (Jacobian map).

② Any $\mathbb{RP}^2 \longrightarrow T^2$ homotopic to const.

$\mathbb{Z}/2 \longrightarrow \mathbb{Z}^2$ (must be trivial)

**Cor.** $K(G,1)$ spaces are unique up to homotopy equiv.

**Pf.** If $X, Y$ are $K(G,1)$'s, the identity $G \to G$ gives

$X \to Y$

$Y \to X$. □

Idea of Pf of Prop Given $f: \pi_1(X) \to G$

Assume with loss of gen. $X$ has one vertex.

0-cells. Send $x_0$ to $y_0$

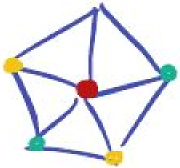
1-cells. Determined by $f$.

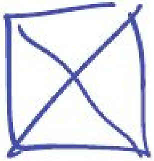
2-cells. Det. by $f$.

uniqueness: since $\tilde{Y} \simeq *$. □

# Application of Cov spaces

**Thm.** A convex polytope made of triangles is 3-colorable $\iff$ even # of triangles at each vertex.

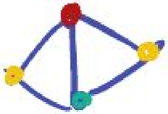




**Pf** (Kontsevich).

For each triangle, color it all 6 ways.

Glue together when sides match and triangles match in original polytype



This is a cov. space. (actually, every connected component is).

Each connected component is trivial cover. □

# Homology

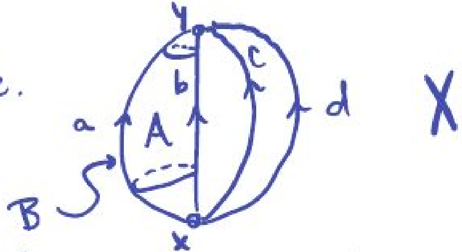
$\pi_1$ is useful, but hard to compute.

$\pi_i$ is harder to compute:

$\pi_m(S^n)$ is a major open problem.

Homology is a computable version...

*Example.*



$C_0$ = free abel gp on $x, y$.

$C_1$ = free abel gp on $a, b, c, d$

$C_2$ = free abel gp on $A, B$.

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$c - d = -d + c$

is the *unbased* clockwise loop around $c$ & $d$.

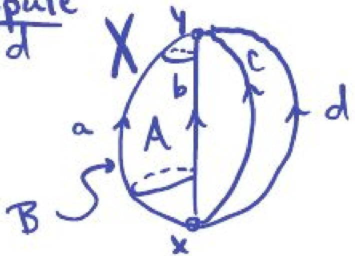
An elt of $H_1(X)$ is a *1-cycle*: an elt of $C_1$ with no boundary.

Two are equiv. if they differ by boundary of elt of $C_2$.

So $H_1(X) = $ 1-cycles / 1-boundaries.

e.g. $a - b = \partial A \Rightarrow a - b \sim 0$.

Compute
$\sim \partial$



X

"boundary map"

$\partial_0 : C_0 \longrightarrow 0$

$\partial_1 : C_1 \longrightarrow C_0$

$a, b, e, d \longmapsto y - x$

$\partial_2 : C_2 \longrightarrow C_1$

$A, B \longmapsto a - b$

So $H_1(X) = $ 1-cycles / 1-boundaries

$= \ker \partial_1 / \operatorname{Im} \partial_2$

exercise →

$= \langle a-b, b-c, c-d \rangle / \langle a-b \rangle$

$\cong \mathbb{Z}^2$

$H_2(X) = \ker \partial_2 / \operatorname{Im} \partial_3$

$= \mathbb{Z}.$

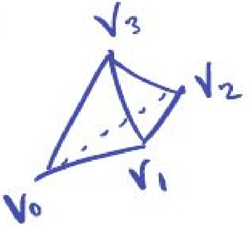
$H_0(X) = \ker \partial_0 / \operatorname{Im} \partial_1$

$= \langle x, y \rangle / \langle y - x \rangle$

$= \langle x, y-x \rangle / \langle y - x \rangle \cong \mathbb{Z}$

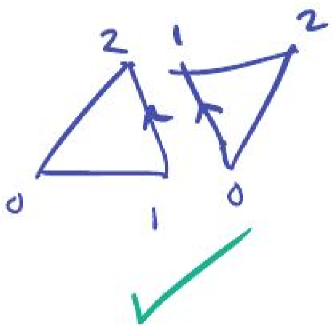
# Δ-complex

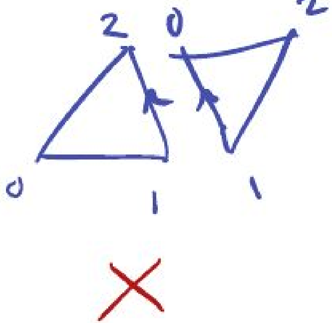
Ordered Simplex :



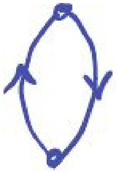
order the vertices on each simplex

A Δ-complex is obtained from a collection of ordered simplices by gluing faces in order preserving way:



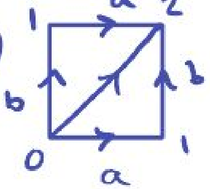


Not a Δ-complex :



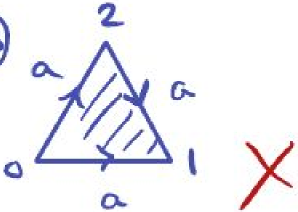
## Examples

①



②

③

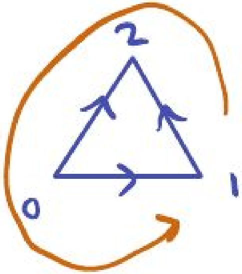


Can subdivide and make it a Δ-complex (exercise).

## Boundaries

$$\partial([v_0,\dots,v_n]) = \sum (-1)^i [v_0,\dots,\hat{v}_i,\dots,v_n]$$

e.g. $\partial([v_0,v_1,v_2]) = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$



exercise: think about $\partial($  $)$.

**Lemma.** $\partial_{n-1} \circ \partial_n = 0$.

*Pf.* Check on each simplex

$$\partial_{n-1}\partial_n([v_0,\dots,v_n]) =$$

$$\partial_{n-1}\left(\sum (-1)^i [v_0,\dots,\hat{v}_i,\dots,v_n]\right)$$

$$= \sum_{j<i} (-1)^{i+j} [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$$

$$+ \sum_{i<j} (-1)^{i+j-1} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$$

$$= 0. \qquad \square$$

In other words: $\operatorname{Im} \partial_n \subseteq \ker \partial_{n-1}$

We now have:

$$\cdots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

where $\Delta_i(X)$ = free abel gp on $i$-simplices

<u>and</u>. $\operatorname{Im} \partial_n \subseteq \ker \partial_{n-1}$.
(prev lemma).

<u>So</u>. It makes sense to define

$$H_k(X) = \ker \partial_{k-1} / \operatorname{Im} \partial_k$$

$$= k\text{-cycles} / k\text{-boundaries}.$$

<u>Examples</u> ① $X = S^1$

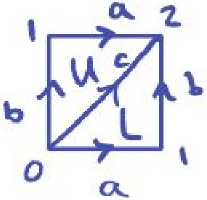
$\Delta_0(X) = \langle v \rangle \cong \mathbb{Z}$

$\Delta_1(X) = \langle e \rangle \cong \mathbb{Z}$

$\partial_1 = 0 \quad \partial_1(e) = v - v = 0.$

$$\Longrightarrow H_k(X) = \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

② $X = T^2$



$\partial_1 = 0.$ $\partial_0 = \partial_3 = 0.$

$\partial_2(u) = \partial_2(L) = a+b-c$

$H_0(X) = \mathbb{Z}/_0 \cong \mathbb{Z}$

$H_1(X) = \langle a,b,c \rangle / _{\langle a+b-c \rangle} \cong \mathbb{Z}^2$

$H_2(X) = \langle u-L \rangle /_0 \cong \mathbb{Z}$

$u-L$ is the torus.

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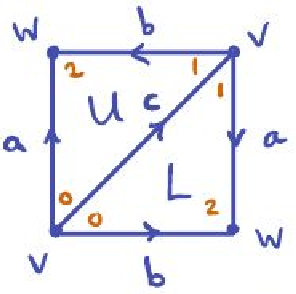
③ $X = \mathbb{R}P^2$



$$\begin{tikzcd} \Delta_2 \arrow[r, "\partial_2"] \arrow[d, phantom, sloped, "\cong"] & \Delta_1 \arrow[r, "\partial_1"] \arrow[d, phantom, sloped, "\cong"] & \Delta_0 \arrow[r, "\partial_0"] \arrow[d, phantom, sloped, "\cong"] & 0 \\ \mathbb{Z}^2 \arrow[d, phantom, sloped, "="] & \mathbb{Z}^3 \arrow[d, phantom, sloped, "="] & \mathbb{Z}^2 \arrow[d, phantom, sloped, "="] & \\ \langle U, L \rangle & \langle a, b, c \rangle & \langle v, w \rangle & \end{tikzcd}$$

$\partial_2$ : $U \longmapsto -a+b+c$

$L \longmapsto a-b+c$

$\partial_1$ : $a \longmapsto w-v$

$b \longmapsto w-v$

$c \longmapsto 0$

$\partial_0$ $v, w \longmapsto 0$

$$H_0(X) = \ker\partial_0 / \operatorname{im}\partial_1$$

$$= \langle v, w \rangle / \langle w-v \rangle = \mathbb{Z}$$

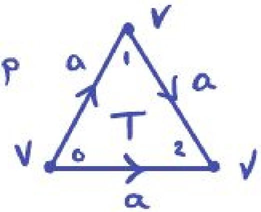
$$H_1(X) = \ker\partial_1 / \operatorname{im}\partial_2$$

$$= \underset{\langle c,\, a-\cancel{b+c} \rangle}{\overset{\phantom{x}}{\langle a-b, c \rangle}} \Big/ \underset{\langle 2c,\, a-\cancel{b}+c \rangle}{\langle -a+b+c,\, a-b+c \rangle}$$

$$= \langle c \rangle / \langle 2c \rangle \cong \mathbb{Z}/2$$

$$H_2(X) = \ker\partial_2 / \operatorname{im}\partial_3 = 0/0 = 0.$$

④ $X$ = Dunce cap



($X$ is contractible but not collapsible)

$$\begin{array}{ccccccc} \Delta_2 & \longrightarrow & \Delta_1 & \longrightarrow & \Delta_0 & \longrightarrow & 0. \\ \| & & \| & & \| & & \\ \langle T \rangle & & \langle a \rangle & & \langle v \rangle & & \end{array}$$

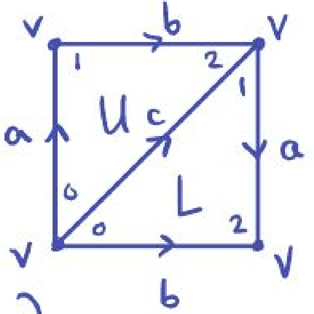
$$T \longmapsto a \longmapsto 0$$
$$v \longmapsto 0$$

$H_0(X) = \langle v \rangle / 0 = \mathbb{Z}$

$H_1(X) = \langle a \rangle / \langle a \rangle = 0$

$H_2(X) = 0/0 = 0.$

⑤ $X$ = Klein bottle.



$5a+7b$
$= 5(a+b)+2b = 5c+2b$

$H_1(X) = \ker\partial_1 / \operatorname{im}\partial_2$

$= \langle a,b,c \rangle / \langle a+b-c, a-b+c \rangle$

What abelian gp is this?
Answer: Smith normal form.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{\text{col op}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow[\text{op}]{\text{row}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$ each diag divides next

$$\rightsquigarrow \mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/0\mathbb{Z} = 0 \times \mathbb{Z}/2 \times \mathbb{Z}$$

Will show: $H_1(X) = \pi_1(X)^{ab}$

Exercise: $H_1(M_g) = \mathbb{Z}^{2g}$

## Exact Sequences

A seq. of homoms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is **exact** if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$

is a **chain complex** if $\ker \alpha_n \supseteq \operatorname{im} \alpha_{n+1}$

i.e. $\alpha_n \circ \alpha_{n+1} = 0$.

### Facts

(i) $0 \to A \xrightarrow{\alpha} B$ exact $\Longleftrightarrow$ $\alpha$ inj.

(ii) $A \xrightarrow{\alpha} B \to 0$ exact $\Longleftrightarrow$ $\alpha$ surj

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ exact $\Longleftrightarrow$ $\alpha$ is $\cong$

(iv) $0 \to A \xrightarrow{\alpha} B \to C \to 0$ exact $\Longleftrightarrow C \cong B/A$

where $A$ is identified with subgp of $B$ by the map $\alpha$

$$\left( \begin{array}{l} 0 \to \mathbb{Z} \xrightarrow{\times 6} \mathbb{Z} \to \mathbb{Z}/6 \to 0 \\ \qquad \Rightarrow \text{"}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}/6\text{"} \end{array} \right).$$

### Four Theorems

① Long exact seq. for collapsing a subcomplex.

② Long ex. seq. for a pair

③ Excision

④ Mayer-Victoris.

## ① Collapsing a Subcomplex

**Thm.** $(X,A)$ is a CW pair

There is an exact seq.

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A)$$

$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \longrightarrow \cdots$$

$$\longrightarrow \tilde{H}_0(X/A) \longrightarrow 0$$

where $i: A \hookrightarrow X$

$q: X \longrightarrow X/A$.

**Cor.** $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{otherwise.} \end{cases}$

**Pf.** Take $X = D^n$
$A = S^{n-1}$ $\rightsquigarrow$ $X/A = S^n$

Induction on $n$.

$\tilde{H}_0(S^0) = \mathbb{Z}$.

Let $n > 0$. By Thm:

$$\cdots \longrightarrow \underset{0}{\cancel{\tilde{H}_i(D^n)}} \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \underset{0}{\cancel{\tilde{H}_{i-1}(D^n)}}$$

$$\Longrightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

# Simplicial Homology

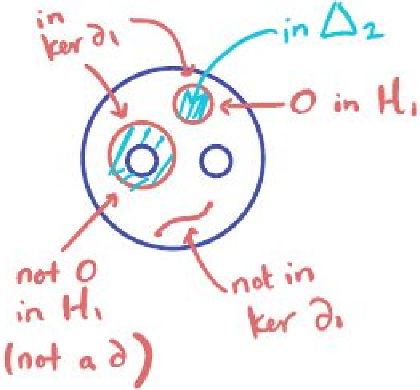
$$\cdots \longrightarrow \Delta_{n+1} \xrightarrow{\partial_{n+1}} \Delta_n \xrightarrow{\partial_n} \Delta_{n-1} \longrightarrow \cdots$$

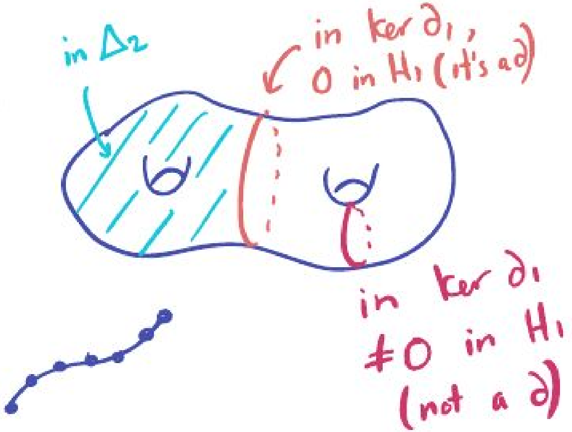
Fact. $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$ (or: $\partial_n \circ \partial_{n+1} = 0$)

$$\rightsquigarrow H_n(X) = {}^{\ker \partial_n}\big/_{\operatorname{Im} \partial_{n+1}}$$

## Four Theorems

1. ① Long exact seq. for collapsing a subcomplex.
2. ② Long ex. seq. for a pair
3. ③ Excision
4. ④ Mayer-Vietoris.





## Singular Homology

Simplicial hom. computable but

① Not obvious that homeomorphic spaces have same homology.

② Hard to prove general thms.

So: A singular $n$-simplex in $X$ is a map $\sigma: \Delta^n \to X$

Let $C_n(X)$ = free abel. gp on these.

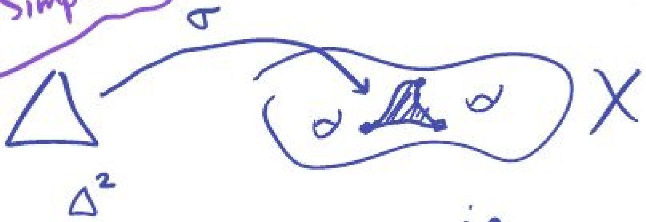
= gp of $n$-chains

$= \{\sum n_i \sigma_i : n_i \in \mathbb{Z}$, $\sigma_i : \Delta^n \to \mathbb{Z}\}$

We'll show: Singular = Simplicial

Boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$

$\sigma \mapsto \sum (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$



Still have: $\partial_{n+1} \circ \partial_n = 0$ i.e. $\mathrm{Im}\, \partial_{n+1} \subseteq \ker \partial_n$

$\rightsquigarrow H_n(X) = \ker \partial_n / \mathrm{Im}\, \partial_{n+1}$.

"singular homology"

Hard to compute. Not obvious that

① $H_k(X) = 0$ $k > \dim X$

② $H_k(X)$ is ever countable.

**Prop.** $X$ = space with path components $X_\alpha$

$$\Rightarrow H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

**Prop.** $X$ = nonempty, path conn $\Rightarrow H_0(X) = \mathbb{Z}$

$X$ has $n$ path comp $\Rightarrow H_0(X) = \mathbb{Z}^n$

**Pf.** By first Prop. suffices to prove 1st statement.

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

↳ free abel. gp on pts in $X$

$$H_0(X) = \ker\partial_0 / \operatorname{Im}\partial_1$$

$$= C_0(X)/\operatorname{Im}\partial_1$$

$\operatorname{Im}\partial_1 = \{v-w : v, w \text{ connected by path}\}$ □

**Prop.** $X = \mathrm{pt} \Rightarrow$

$$H_i(X) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$$

**Pf.** $C_n(X) \cong \mathbb{Z} \quad \forall n.$

$$\partial(\sigma_n) = \sum (-1)^i \sigma_{n-1}$$

$$= \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even.} \end{cases}$$

$$C_4 \to C_3 \to C_2 \to C_1 \to C_0$$

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \boxed{\mathbb{Z}} \to 0$$

↳ $\ker = \mathbb{Z} / \operatorname{Im}\mathbb{Z} = 0$

↳ $\ker = 0$

$\ker/\operatorname{Im} = \mathbb{Z}$ □

## Reduced Homology

$$\tilde{H}_n(X) = \begin{cases} H_n(X)/\mathbb{Z} & n=0 \\ H_n(X) & n>0 \end{cases}$$

Can achieve this by replacing the last 0 map above by "evaluation map" $\varepsilon$

$$C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\varepsilon\left(\sum n_i v_i\right) = \sum n_i$$

$\rightsquigarrow$ same $\tilde{H}$ as above.

## Homotopy Invariance

Goal: $f : X \to Y \rightsquigarrow f_* : H_n(X) \to H_n(Y)$

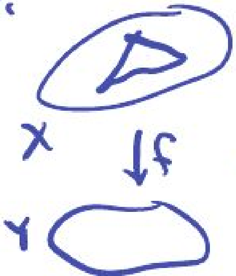
& $f$ a hom. eq. $\Rightarrow$ $f_*$ an isom.

First: $f \rightsquigarrow f_\# : C_n(X) \to C_n(Y)$

$$\sigma \longmapsto f \circ \sigma$$

Have: $f_\# \partial = \partial f_\#$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$



$f_\#$ called a chain map. It takes cycles to cycles, $\partial$ to $\partial$

$\rightsquigarrow$ induced map

$$f_* : H_n(X) \to H_n(Y).$$

Facts. $(fg)_* = f_* g_*$

$id_* = id$

Thm. $f, g : X \to Y$ homotopic

$\Longrightarrow f_* = g_*$

Cor. $f : X \to Y$ hom eq

$\Longrightarrow f_*$ is $\cong$

Example. $X \simeq * \Longrightarrow H_*(X) = 0$

## ① Collapsing a Subcomplex

**Thm.** $(X, A)$ is a CW pair

There is an exact seq.

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0$$

where $i : A \hookrightarrow X$

$q : X \longrightarrow X/A$.

To prove the Thm will do something more general.

**Cor.** $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{otherwise.} \end{cases}$

**Pf.** Take $X = D^n$, $A = S^{n-1}$ $\rightsquigarrow$ $X/A = S^n$

Induction on $n$.

$\tilde{H}_0(S^0) = \mathbb{Z}$.

Let $n > 0$. By Thm:

$$\cdots \longrightarrow \underbrace{\cancel{\tilde{H}_i(D^n)}}_{0} \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \underbrace{\cancel{\tilde{H}_{i-1}(D^n)}}_{0}$$

$$\Longrightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

# Homotopy Invariance

Goal: $f: X \to Y \rightsquigarrow f_*: H_n(X) \to H_n(Y)$

& $f$ a hom. eq. $\Rightarrow f_*$ an isom.

First: $f \rightsquigarrow f_\#: C_n(X) \to C_n(Y)$

$$\sigma \longmapsto f \circ \sigma$$

Have: $f_\# \partial \cdot \partial f_\#$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \overset{c}{C_{n+1}(X)} \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & \overset{0}{C_n(X)} \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & \overset{f_\#(c)}{C_{n+1}(Y)} \arrow[r, "\partial"] & \overset{0}{C_n(Y)} \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

$f_\#$ called a chain map. It takes cycles to cycles, $\partial$ to $\partial$

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$\rightsquigarrow$ induced map

$$f_*: H_n(X) \to H_n(Y).$$

**Facts.** $(fg)_* = f_* g_*$

$id_* = id$

**Thm.** $f, g: X \to Y$ homotopic

$\Rightarrow f_* = g_*$

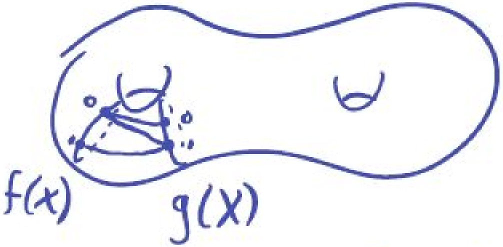
**Cor.** $f: X \to Y$ hom eq

$\Rightarrow f_*$ is $\cong$

**Thm.** $f, g : X \to Y$ homotopic

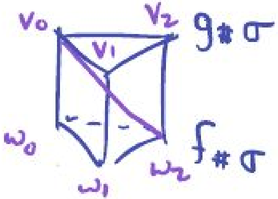
$$\Longrightarrow f_* = g_*$$

**Idea.** $X = S^1 \quad Y = M_2$



$f_* = g_*$ means $f_*(S^1) - g_*(S^1)$ is a boundary. In the example, it is the boundary of a cylinder.

**Pf.** For $\sigma$ any singular $n$-simplex in $X$. The homotopy gives a prism in $Y$



Label the vertices on top by $v_0, \dots, v_n$

bot by $w_0, \dots, w_n$

$\Delta^n \times I$ decomposes as sum of

$$[v_0, \dots, v_i, w_i, \dots, w_n]$$

Call this sum $P(\sigma)$

Check $\partial(\Delta^n \times I)$ = top − bottom.

$\blacksquare$

Algebraically: We just defined a map

$$P : C_n(X) \to C_{n+1}(Y)$$
$$\sigma \longrightarrow P(\sigma)$$

with $\partial P = \underset{\text{top}}{g_\#} - \underset{\text{bot}}{f_\#} - \underset{\text{sides}}{P\partial}$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \overset{c}{C_{n+1}(X)} \arrow[r, "\partial"] \arrow[d, "f_\#"] & \overset{0}{C_n(X)} \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & \overset{f_\#(c)}{C_{n+1}(Y)} \arrow[r, "\partial"] & \overset{0}{C_n(Y)} \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Chain homotopy.

The thm follows:

If $\alpha \in C_n(X)$ is a cycle then

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\cancel{\partial(\alpha)}$$

$\Longrightarrow (g_\# - f_\#)(\alpha)$ is a $\partial$

$\Longrightarrow g_*(\alpha) = f_*(\alpha)$.

□

four goals

① Contracting subcomplex

①' Long ex seq for a pair

② Excision

③ Mayer-Vietoris.

## ① Collapsing a Subcomplex

**Thm.** $(X, A)$ is a CW pair

There is an exact seq.

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{q_*} \tilde{H}_{n-1}(X/A) \cdots\cdots \xrightarrow{q_*} \tilde{H}_0(X/A) \longrightarrow 0$$

where $i : A \hookrightarrow X$

$q : X \longrightarrow X/A$.

$\partial = ?$

**Cor.** $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{otherwise.} \end{cases}$

**Pf.** Take $X = D^n$, $A = S^{n-1}$ $\rightsquigarrow$ $X/A = S^n$

Induction on $n$.

$\tilde{H}_0(S^0) = \mathbb{Z}$.

Let $n > 0$. By Thm:

$$\cdots \longrightarrow \underbrace{\cancel{\tilde{H}_i(D^n)}}_{0} \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \underbrace{\cancel{\tilde{H}_{i-1}(D^n)}}_{0}$$

$$\Longrightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

## Application

### Brouwer Fixed Pt Thm

Any map $f: D^n \to D^n$ has a fixed pt

Pf. Assume $f$ has no fixed pt

→ retraction $r: D^n \to S^{n-1}$

retraction means:

$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$$ compos. is id

$\Longrightarrow r_* i_* = \mathrm{id}$.

but $\tilde{H}_{n-1}(D^n) = 0$ contrad. □

Same as $n=2$ case with $\pi_1$ replaced by $H_{n-1}$.

Will prove an alternate version of the theorem first: ①′

**Relative Homology**

$A \subseteq X \rightsquigarrow C_n(X,A) = C_n(X) / C_n(A)$

Since $\partial$ takes $C_n(A)$ to $C_{n-1}(A)$ have:

$$\cdots \longrightarrow C_n(X,A) \longrightarrow C_{n-1}(X,A) \longrightarrow \cdots$$

$\rightsquigarrow$ relative homology gps $H_n(X,A)$.

Elts of $H_n(X,A)$ are relative cycles:

Will show: $H_n(X,A) \cong H_n(X/A)$

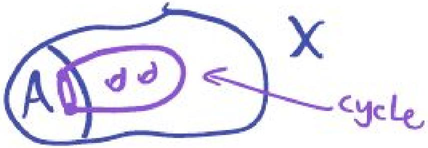
Thm 1′. Long ex. seq.

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \longrightarrow \cdots$$

Proof is "diagram chasing"

**Thm 1'.** Long ex. seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$



$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
& C_{n+1}(A) \arrow[r, "\partial"] \arrow[dd, "i"'] & C_n(A) \arrow[r, "\partial"] \arrow[dd, "i"'] & C_{n-1}(A) \arrow[dd, "i"'] \\
& & \partial(\tilde{c}) \arrow[u] & \\
\tilde{c} & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_{n-1}(X) \arrow[d, "q"'] \\
& C_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d] \arrow[uuur, dashed, bend right] & C_n(X,A) \arrow[r, "\partial"] \arrow[d, "0"'] \arrow[uuur, dashed, bend right] & C_{n-1}(X,A) \arrow[d] \\
\begin{matrix} c \\ \text{cycle} \end{matrix} & 0 & 0 & 0
\end{tikzcd}$$

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**Thm 1'.** Long ex. seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

(represents c (which represents [c]))

$[a] := \partial[c]$

"diagram chasing"

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_{n-1}(A) \arrow[d, "i"'] \\
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_{n-1}(X) \arrow[d, "q"'] \\
C_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d] \arrow[ruu, dashed, bend right] & C_n(X,A) \arrow[r, "\partial"] \arrow[d] \arrow[ruu, dashed, bend right] & C_{n-1}(X,A) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

$a$

$\partial\tilde{c} \in \ker q$

$\quad = i(a)$

$\tilde{c}$

$c$

$[c]$

$0$

$$\begin{tikzcd}
 & \overset{a'}{C_n(A)} \arrow[r, "\partial"] \arrow[d, "i"'] & C_{n-1}(A) \arrow[d, "i"'] & {[a] := \partial[c]} \quad {\overset{a''}{\underset{a+\partial(a')}{\shortparallel}}} \\
 & & & a = \partial\tilde{c} \\
\tilde{c} & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"', "\tilde{c}'"] & C_{n-1}(X) \arrow[d, "q"'] & \partial\tilde{c} \in \ker q \\
 & & & \overset{\shortparallel}{i(a)} \\
 & \underset{c}{C_n(X,A)} \arrow[r, "\partial"] \arrow[uuuur, dashed, bend right] & \underset{0}{C_{n-1}(X,A)} &
\end{tikzcd}$$

Claim: $\partial$ is a well-def homom.

Two choices:
- $c$ in $[c]$
- $\tilde{c}$

We'll check that choice of $\tilde{c}$ doesn't matter.

Say $\tilde{c}'$ another choice...

Then $q(\tilde{c}) = q(\tilde{c}')$ or:

$$\tilde{c}' = \tilde{c} + i(a')$$

Instead of $\partial\tilde{c}$ we get

$$\partial\tilde{c}' = \partial\tilde{c} + \partial i(a')$$

$$= \partial\tilde{c} + i\partial(a')$$

This is homologous to $a$ since

$$\partial(a') = 0 \text{ in } H_{n-1}(A).$$

You: check choice of $c$ & homom.

**Thm 1'.** Long ex. seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A)$$
$$\xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

**Pf.** More diagram chasing.
6 containments to check.

- $\operatorname{Im} \partial \subseteq \ker i_*$ i.e. $i_* \partial = 0$.
  $i_* \partial$ takes $[c]$ to $[\partial \tilde{c}] = 0$.
- $\ker i_* \subseteq \operatorname{Im} \partial$ : Say $a \in C_{n-1}(A)$
  $a \in \ker i_* \Longrightarrow i(a) = \partial c \quad c \in C_n(X)$
  $\Longrightarrow q(c)$ is a cycle (its $\partial$ is in $A$)*
  and $\partial[q(c)] = a$

* $\partial q(b) = q \partial(b) = q(a) = 0$ $\square$

Some facts about relative hom:

**Prop.** $H_n(X,A) = 0 \ \forall n \Longleftrightarrow H_n(A) = H_n(X) \ \forall n$

Reduced relative homology makes sense

$\rightsquigarrow \tilde{H}_n(X,A) = H_n(X,A)$ if $A \neq \emptyset$.

**Prop.** If $f, g : (X,A) \to (Y,B)$ homotopic then $f_* = g_*$

More: For a triple $B \subseteq A \subseteq X$

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A)$$
$$\to H_{n-1}(A,B) \to \cdots$$

Then spectral sequences...

<u>Next</u>

<u>Thm 2</u>. $A, B \subseteq X$

Mayer-Vietoris

interiors of $A, B$ cover $X$

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X)$$
$$\to H_{n-1}(A \cap B) \to \cdots$$

"Van Kampen for Homology"

<u>Example</u>. $S^n = X$ $A = B = D^n$

$A \cap B = S^{n-1}$

$\rightsquigarrow H_n(S^n)$.

Mar 4

# Applications of Homology

① Jordan curve thm.

**Thm.** Let $h : S^1 \to \mathbb{R}^2$ be an embedding (injective)

Then $\mathbb{R}^2 \setminus h(S^1)$ has exactly 2 connected components.

Osgood curves...

(b) also implies $H_1(S^3 \setminus \text{knot}) = \mathbb{Z}$

**Prop.** (a) If $h : D^k \to S^n$ is an embedding then

$$\tilde{H}_i(S_n \setminus h(D^k)) = 0 \ \forall i.$$

(b) If $S^k \to S^n$ is embedd. $k<n$

$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{o.w.} \end{cases}$$

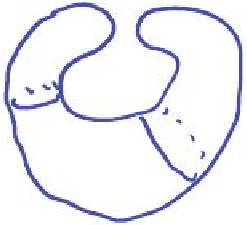
---

(b) implies any $S^{n-1}$ in $S^n$ divides $S^n$ into 2 components, each with the homology of a pt.

For $n=2$: Jordan curve

For $n=3$: It is possible for one component to not be simply conn.

# Alexander Horned Spheres



etc.

Outside has $\pi_1$ $\infty$ gen. but by Prop. $H_1 = 0$.

etc.

intersect these.

$$\pi_1(\text{outside}) = \langle \alpha_0, \alpha_1, \ldots \mid [\alpha_1, \alpha_2] = \alpha_0, \; [\alpha_3, \alpha_4] = \alpha_1, \; [\alpha_5, \alpha_6] = \alpha_2, \ldots \rangle$$

**Prop.** (a) If $h: D^k \to S^n$ is an embedding then

$$\tilde{H}_i(S_n \setminus h(D^k)) = 0 \ \forall i.$$

(b) If $S^k \to S^n$ is embedd. $k<n$

$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i=n-k-1 \\ 0 & \text{o.w.} \end{cases}$$

---

ⓐ Induct on $k$.

$k=0$ : $S^n - h(D^k) \cong \mathbb{R}^n$ ✓

Replace $D^k$ with $I^k$

half-cube

Let $A = S^n - h(I^{k-1} \times [0, 1/2])$

$B = S^n - h(I^{k-1} \times [1/2, 1])$

$A \cup B = S^n - h(I^{k-1} \times \{1/2\})$

Induction $\Rightarrow \tilde{H}_i(A \cup B) = 0$.

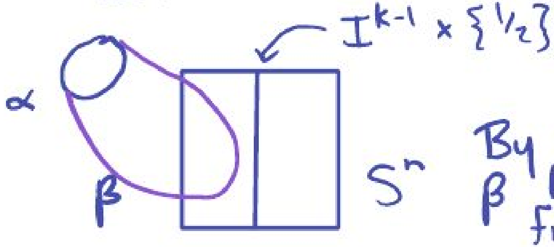
Mayer-Vietoris

$$\Phi : \tilde{H}_i(\underbrace{A \cap B}_{S^n - h(I^k)}) \xrightarrow{\cong} \tilde{H}_i(A) \oplus \tilde{H}_i(B)$$

Assume for contrad $[\alpha] \neq 0$ in $\tilde{H}_i(A \cap B)$

Then $\Phi$ of $[\alpha] \neq 0$ in $\tilde{H}_i(A)$ $A = S^n -$ half cube.

But $\alpha = 0$ in $\tilde{H}_i(A \cap B) = 0$. repeat!



$\Rightarrow \alpha = \partial \beta$

$\beta \subseteq S^n - I^{k-1}$

By compactness, $\beta$ lives in some finite stage.

**Prop.** (a) If $h: D^k \to S^n$ is an embedding then

$$\tilde{H}_i(S_n \setminus h(D^k)) = 0 \quad \forall i.$$

(b) If $S^k \to S^n$ is embedd. $k<n$

$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i=n-k-1 \\ 0 & \text{o.w.} \end{cases}$$

**Proof of ⓑ** Induct on k.

& MV.

**Exercise.** The case $k=n$.

$\rightsquigarrow$ $S^n$ cannot embed in $\mathbb{R}^n$

$\mathbb{R}^m$ cannot embed in $\mathbb{R}^n$ $m>n$.

## ② Invariance of domain

**Thm.** $U$ open in $\mathbb{R}^n$

$h: U \to \mathbb{R}^n$ embedding

$\Longrightarrow h(U)$ open in $\mathbb{R}^n$.

**Cor.** $M$ = compact $n$-manifold

$N$ = connected $n$-man

Any embedding $M \to N$ is surjective, hence a homeo.

**Thm.** $U$ open in $\mathbb{R}^n$

$h: U \to \mathbb{R}^n$ embedding

$\Rightarrow h(U)$ open in $\mathbb{R}^n$.

**Pf.** Think of $\mathbb{R}^n$ as $S^n \setminus pt$

Enough to show $h(U)$ open in $S^n$

Let $x \in U$, $D^n$ = disk about $x$ in $U$

Suffices to show $h(\text{int } D^n)$ open in $S^n$.

Prop (b) $\Rightarrow$ $S^n \setminus h(\partial D^n)$ has 2 path comp.

The two path components are:

$h(\text{int } D^n)$

$S^n \setminus h(D^n)$

you: justify

$h(\partial D^n)$ is closed (compact in Hausdorff)

$\Rightarrow$ $S^n \setminus h(\partial D^n)$ open.

$\Rightarrow$ path components are the conn. components. (true in a loc. comp. space)

**Fact.** An open set with finitely many components has each component as an open subspace.

$\Rightarrow$ $h(\text{int } D^n)$ open in $S^n \setminus h(\partial D^n)$

$\Rightarrow$ open in $S^n$. □

Mar 7

## ② Invariance of domain

**Thm.** $U$ open in $\mathbb{R}^n$

$h: U \to \mathbb{R}^n$ embedding

$\Rightarrow h(U)$ open in $\mathbb{R}^n$.

**Cor.** $M$ = compact $n$-manifold*

$N$ = connected $n$-man

Any embedding $M \to N$

is surjective, hence a homeo.

*every pt has a nbd homeo to $\mathbb{R}^n$. + Haus.

**Pf of Cor.**

$h(M)$ closed in $N$ (compact in Haus.)

Since $N$ connected, suffices to show

$h(M)$ open in $N$.

Let $x \in M$, Choose nbd $V \cong \mathbb{R}^n$

of $h(x) \in N$.

Choose open nbd $U$ of $x$ in $\overset{\text{open}}{\overset{\shortparallel}{h^{-1}(V)}}$

homeo to $\mathbb{R}^n$.

But $h|U$ is an embedding (restr. of embed)

$\Rightarrow h(U)$ open by Thm. $\Rightarrow h(M)$ open

$\uparrow$ in $V$, hence $N$.

□

## ③ Division Algebras

An algebra over $\mathbb{R}$ is $\mathbb{R}^n$ with bilin. mult.

$a(b+c) = ab + ac$ etc...

It's a division alg if $ax = b$ always solvable for $a \neq 0$ (no zero divisors).

Examples. $\mathbb{R}$, $\mathbb{C}$

Thm. $\mathbb{R}, \mathbb{C}$ are only finite dim algebras over $\mathbb{R}$ that are commutative and have id.

Pf that any such alg has dim $\leq 2$:

Define $f: S^{n-1} \to S^{n-1}$ by $f(x) = x^2/|x^2|$

$\rightsquigarrow$ induced map $\bar{f}: \mathbb{RP}^{n-1} \to S^{n-1}$

well def: no 0 divisors!

Claim $\bar{f}$ injective.

$\alpha = \left(\frac{|x^2|}{|y^2|}\right)^{1/2}$

Pf $\bar{f}(x) = \bar{f}(y) \Rightarrow x^2 = \alpha y^2$

$$\Rightarrow x^2 - \alpha^2 y^2 = 0 \xRightarrow{\text{commut!}} (x + \alpha y)(x - \alpha y) = 0$$

$$\xRightarrow{\text{no 0 div}} x = \pm \alpha y \Rightarrow x = y \text{ in } \mathbb{RP}^{n-1} \quad \square$$

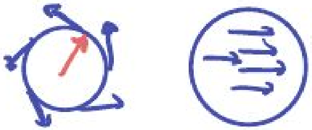
$\bar{f}$ inj on compact Havs. → embedding.

Cor ⇒ $\bar{f}$ homeo. $\Rightarrow n \leq 2$. (use $\pi_1$ or $H_1$ or $H_{n-1}$)

Some more algebra to finish the thm. □

## ④ Hairy ball thm (Can't comb a monkey)

**Thm.** $S^n$ has a continuous field of nowhere 0 tangent vectors iff $n$ odd.



$n$ odd: $v(x_1,\dots,x_{2k})$

$= (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$

**Tool: Degree**

$$f: S^n \longrightarrow S^n \rightsquigarrow f_*: \overset{\mathbb{Z}}{\overset{\shortparallel}{H_n(S^n)}} \longrightarrow H_n(S^n)$$

$$\alpha \longmapsto d\alpha$$

$d$ = degree of $f$.

**Facts.**

(i) $\deg \mathrm{id} = 1$

(ii) $\deg f = 0$ if $f$ not surj.

(iii) $\deg f = \deg g \Leftrightarrow f \simeq g$.

$\Rightarrow$ Hopf.

(iv) $\deg fg = \deg f \deg g$.

(v) $\deg f = -1$ for $f$ a reflection thru equator.

(vi) $\deg(\text{antipodal}) = (-1)^n$.

**Thm.** $S^n$ has a continuous field of nowhere 0 tangent vectors iff $n$ odd.

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$\Rightarrow$ Hopf.

(iv) $\deg fg = \deg f \deg g$.

(v) $\deg f = -1$ for $f$ a reflection thru equator.

(vi) $\deg(\text{antipodal}) = (-1)^{n+1}$.

**Pf.** ($\Rightarrow$) Let $v(x)$ = vect. field on $S^n$.

$\rightsquigarrow$ $v(x) \perp x$ in $\mathbb{R}^{n+1}$.

$v(x) \neq 0 \ \forall x \Rightarrow$ can replace $v(x)$ with $v(x)/|v(x)|$

$\Rightarrow (\cos t)x + (\sin t)v(x)$ is a unit $S^1 \subseteq S^n$ in $xv(x)$ plane

is a homotopy from id $(t=0)$ to antipodal map $(t=\pi)$

(iii) $\Rightarrow$ $\deg \mathrm{id} = \deg \text{antip}$.

(vi) $\Rightarrow$ $n$ odd. $\square$

One more fact

(vi) If $f$ has no fixed pts

$$\deg f = (-1)^{n+1}$$

proof: find homotopy to antip. map.
(straight line)

⑤ **Prop.** $\mathbb{Z}/2$ is only gp that acts freely on $S^n$ if $n$ even.

**Pf.** Say $G \curvearrowright S^n$

$$\rightsquigarrow d : G \to \overset{\mathbb{Z}/2}{\overset{\shortparallel}{\{\pm 1\}}} \text{ homom by (iv).}$$

$$\text{Action free} \underset{\text{by (vi)}}{\Rightarrow} d(g) = (-1)^{n+1} \quad \forall g \neq id.$$

$n$ even $\Rightarrow$ $\ker d = 1$

$\Rightarrow G \cong \mathbb{Z}/2$ or $1$

$\square$

Mar 9

## ⑥ Borsuk-Ulam Thm

Thm. $g : S^n \to \mathbb{R}^n$

$\Longrightarrow \exists\ x$ s.t. $g(x) = g(-x)$

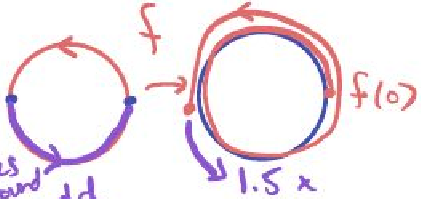
Prop. $f : S^n \to S^n$ odd $\begin{matrix} f(-x) \\ = -f(x) \end{matrix}$

$\Longrightarrow \deg f$ odd

Pf of Prop $(n=1)$

WLOG $f(0) = 0$

$\Rightarrow f(\pi) = \pi$



$f([0,\pi])$ goes $k/2$ times around, $k$ odd

vs. $f([\pi, 0])$ goes $k/2$ times around in same dir. □

Pf of Prop for $n > 1$ uses $\mathbb{Z}/2$ coeffs & transfer homoms.

Pf of Thm. Let $f(x) = g(x) - g(-x)$ odd!

Say $f(x) \neq 0\ \forall\ x$.

Then can define $h(x) = f(x)/|f(x)|$

$h : S^n \to S^{n-1}$

$h|\text{equator} : S^{n-1} \to S^{n-1}$ still odd

Prop $\Longrightarrow h|eq$ odd degree.

But $\deg h|eq = 0$

(it is $\simeq$ const since eq $\simeq$ pt in $S^n$) □

# ⑦ Lefschetz Fixed Pt Thm

For $\varphi : A \to A$  $A$ = fin gen abel gp

$$\operatorname{tr} \varphi = \operatorname{tr}\left( A/\text{torsion} \longrightarrow A/\text{torsion} \right)$$
$$= \operatorname{tr}\left( \mathbb{Z}^k \longrightarrow \mathbb{Z}^k \right)$$

---

$X$ = space with fin gen homology e.g. finite CW complex. of dim $n$

$f : X \longrightarrow X$

The Lefschetz # of $f$ is

$$\tau(f) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}\left( f_* : H_i(X) \to H_i(X) \right)$$

If $f(p) = p$ (fixed pt)

deg $p$ is the degree of

$$\bar{f} : H_n(X, X-p) \longrightarrow H_n(X, X-p)$$

example: ① If $f$ rotates about $p$

deg $p = 1$

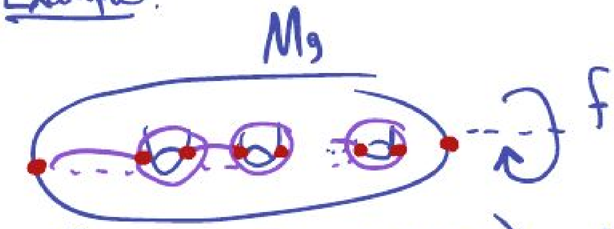
② $f(z) = z^2$  $X = \mathbb{C}$  $\deg 0 = 2$.

<u>Thm</u>. $\tau(f) = \sum_{f(p)=p} \deg(p)$

<u>Cor</u>. Brouwer FPT. $\tau(f) = 1$

**Thm.** $\tau(f) = \sum_{f(p)=p} \deg(p)$

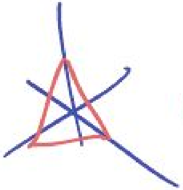
**Example.**

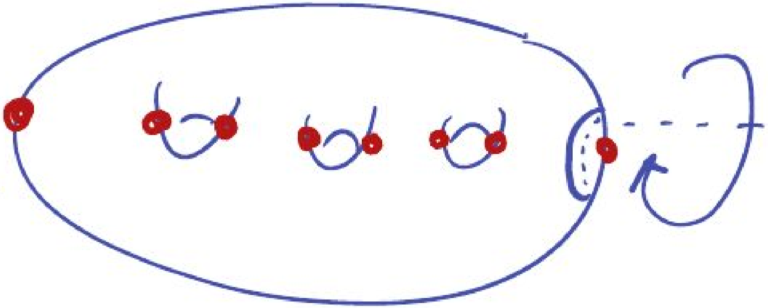


$$\mathrm{tr}\left(f_* : H_0(M_g) \to H_0(M_g)\right) = 1$$
$$\mathrm{tr}\left(f_* : H_1(M_g) \to H_1(M_g)\right) = -2g$$
$$\mathrm{tr}\left(f_* : H_2(M_g) \to H_2(M_g)\right) = 1$$

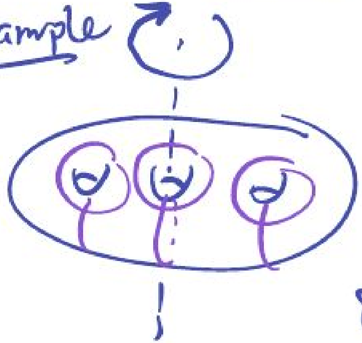
$\tau(f) = 2g+2.$

There are $2g+2$ fixed pts, all of deg 1





**Example.**



$\tau(f) = 1-2+1$
$= 0$

& no fixed pts.

**Cor.** Any linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ n odd has a real eigenvector.

**Pf.** $L$ induces $\mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$. Lefschetz $\overset{\tau(f)=1}{\Longrightarrow}$ fixed pt

## ⑧ Euler characteristic.

$X$ = CW complex

$c_n$ = # $n$ cells

$$\chi(X) = \sum (-1)^i c_i$$

<u>Thm</u>. This is indep of cell decomp.

Even better:

$$\chi(X) = \sum (-1)^i \underbrace{\text{rk}\, H_i(X)}_{\text{rank of } H_i(X)/\text{torsion.}}$$

$C_n$ = usual simplicial/cellular chain complex.

Thm 2.44 in Hatcher.

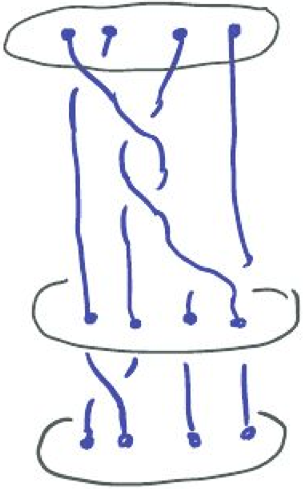
<u>Example</u>. $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$

Mar 11

# Braid Groups

Artin 1925, Hurwitz 1891

$B_n$ = {braids on n strands}/homotopy



$\longmapsto (3\,4) \in S_4$

Multiplication: stacking.

Identity: | | | | |

Inverse: Mirror vertically.

Generators:

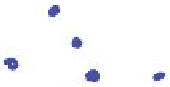
$\sigma_i$

A map $B_n \twoheadrightarrow S_n$.

Kernel: $PB_n$ pure braid group.

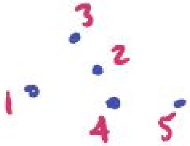
## 2nd interpretation

$C_n$ = config sp. of n distinct pts in $\mathbb{R}^2$



a pt in $C_5$

$U_n$ = config sp. of n labeled distinct pts in $\mathbb{R}^2$



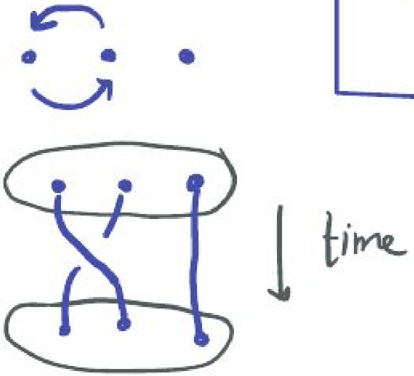
a pt in $U_5$

$U_n = (\mathbb{R}^2)^n \setminus$ big diag.

$\pi_1(C_n) \cong B_n$

$\pi_1(U_n) \cong PB_n$.

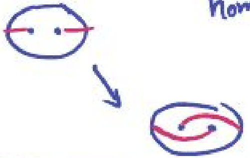
A loop in $C_3$:



### 3rd interp

$B_n \cong \mathrm{Homeo}(D_n, \partial D_n) / \text{homotopy}$

id on $\partial$



So $B_n$ encodes motions of pts (robotics, physics...)

$C_n = \mathrm{Poly}_n$ = space of deg n. square free polynomials

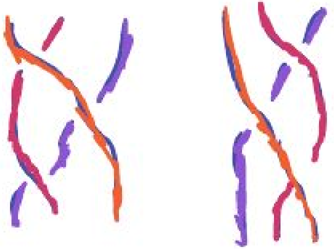
## Some braid gps

$B_1 = 1$

$B_2 \cong \mathbb{Z}$

A $K(G,1)$ space is a space with $\pi_1 = G$ & contractible univ. cover

$B_3$ more complicated.



braid relation

$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

$SL_2\mathbb{Z}$ = $2\times 2$ $\mathbb{Z}$ matrices of det 1

but $B_3/\text{center} \cong PSL_2\mathbb{Z}$

$B_n$ $n>3$ even more complicated...

**Thm.** $B_n$ is torsion free.

(Only elt of finite order is id).

$b^3$ is central in $B_3$.
(exercise).

**Pf outline.** ① $C_n$ is a $\overset{2n\text{-dim}}{K(B_n,1)}$

② If $G$ has torsion, any $K(G,1)$ is $\infty$ dim.

① $C_n$ is a $K(B_n, 1)$

First $U_n \to C_n$ is the covering space. corr to $PB_n$.

To see this: $S_n \curvearrowright U_n$.

$$U_n / S_n = C_n$$

This is a covering sp action.

○ $\odot_{x \in U_n}$ (exercise)

○ ○

So: $U_n$ & $C_n$ have same univ cover.

So: Will show $\tilde{U}_n$ is contractible.

<u>Whitehead's Thm</u> $X$ CW complex

Then $\tilde{X}$ contractible iff $\pi_i(X) = 0$ $i > 1$

preim. of pt.

We have:

$$\begin{tikzcd} \mathbb{R}^2 - \{\substack{n-1 \\ \text{pts}}\} \arrow[r] \arrow[d, equal] & U_n \arrow[d] \\ \text{graph} \cong X_{n-1} & U_{n-1} \end{tikzcd}$$

fiber bundle.

$\rightsquigarrow$ LES for fiber bundles:

0 by ind.

$$\pi_i(\overset{0}{X_{n-1}}) \to \pi_i(U_n) \to \pi_i(U_{n-1}) \to \pi_{i-1}(X_{n-1})\, 0$$

$$\Rightarrow \pi_i(U_n) = 0. \qquad \square$$

② $G$ has torsion $\Rightarrow$ Any $K(G,1)$ is $\infty$ dim.

We'll Show any $K(\mathbb{Z}/2,1)$ is $\infty$-dim

Our fave $K(\mathbb{Z}/2,1)$ : $S^\infty/(\mathbb{Z}/2) = \mathbb{RP}^\infty$.

The chain complex for $H_*(\mathbb{RP}^\infty)$

$$\cdots \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \cdots$$

$\uparrow H_n = \mathbb{Z}/2$ $\uparrow H_{n-1} = 0$.

$\Rightarrow$ $\mathbb{RP}^\infty$ has nontrivial homology in $\infty$ many dim.

$\Rightarrow$ any $K(\mathbb{Z}/2,1)$ is $\infty$ dim.

Similar for $\mathbb{Z}/n$

If $G$ has torsion then

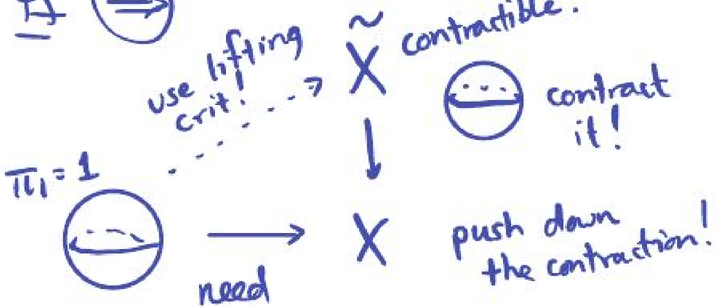
$K(\mathbb{Z}/n,1)$ is a cover of $K(G,1)$

If $K(G,1)$ finite dim then $K(\mathbb{Z}/n,1)$ fin dim

contrad.

# Idea of Whitehead's thm

Pf ⇒



need to show homotopic to pt.

⇐ lift all contractions of spheres + $\pi_1(\tilde{X}) = 1$ by defn.

# Fiber bundles

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

fiber $F$

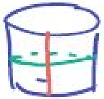
total space $E$

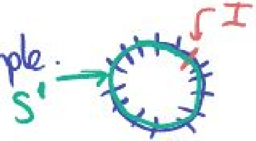
base space $B$

all $\pi^{-1}(b)$ homeo to $F$

and $\forall\, b \in B\ \exists$ nbd $U$ s.t. $\pi^{-1}(U) \cong U \times F$ "locally a product"

example.





$$\begin{tikzcd} I \arrow[r] & A \arrow[d] \\ & S^1 \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r] & \text{Mobius} \arrow[d] \\ & S^1 \end{tikzcd}$$

$A = S^1 \times [-1,1]$

$\pi$ = proj to $S^1 \times \{0\}$

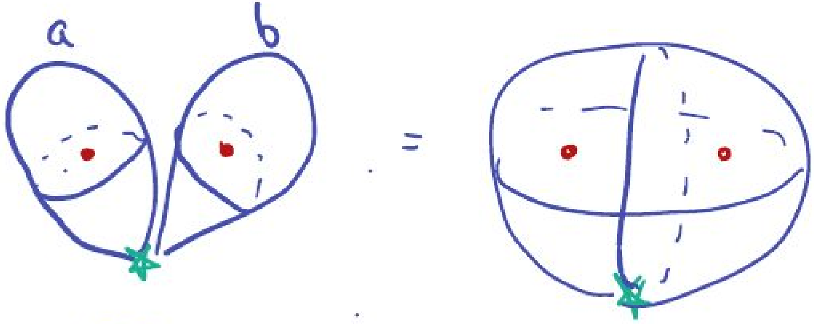
Example Cov. sp $F$ = discrete set

## Homotopy gps

$X = \mathbb{R}^3 - \text{two pts}$

$\pi_1(X) = 1$

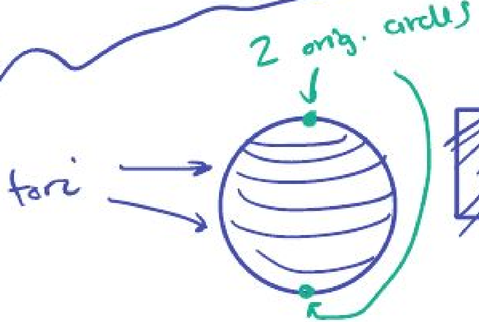
$\pi_2(X) = \mathbb{Z}^2$

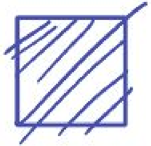


$\pi_i$ abelian

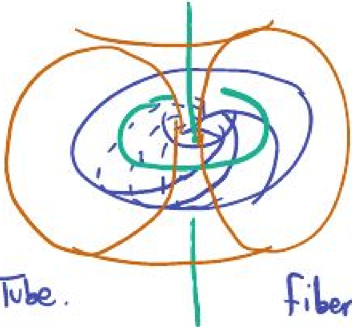
$i > 1$

Hopf fibration is an interesting elt of $\pi_3(S^2)$





See YouTube.



Fill $S^3$ by tori

Fill each torus with (1,1) curves

$S^3 \longrightarrow \text{set of circles}$ $\overset{\shortparallel}{S^2}$

fiber $S^1$

Mar 14

# COHOMOLOGY

Same basic info as homology but:

- mult. structure.
- pairing with homology
- contravariance.

Quick idea: $X = \Delta$ complex

$G = \mathbb{Z}$ or $\mathbb{Z}/2$ or another abel. gp.

$\Delta^i(X)$ = functions from $i$-simplices of $X$ to $G$.

= homoms $\Delta_i(X) \to G$

$$\delta : \Delta^i(X,G) \to \Delta^{i+1}(X,G)$$
$$f \mapsto \delta f$$

for $f \in \Delta^i$, $\sigma$ = $(i+1)$-simplex

$$\delta f(\sigma) = \Sigma(-1)^k f(\partial_k \sigma)$$

$H^*(X,G)$ = homology of this chain complex.

$\Delta^i(X)$ = functions from $i$-simplices of $X$ to $G$.

= homoms $\Delta_i(X) \to G$

$\delta : \Delta^i(X,G) \to \Delta^{i+1}(X,G)$

$f \mapsto \delta f$

for $f \in \Delta^i$, $\sigma = (i+1)$-simplex

$\delta f(\sigma) = \Sigma(-1)^k f(\partial_k \sigma)$

---

Graphs. $X$ = 1-dim $\Delta$-complex

= oriented graph.

Let $f \in \Delta^0(X,G)$

$\delta f(e) = f(\text{end of } e) - f(\text{start of } e)$

= change of $f$ on $e$ "derivative"

Think of $f$ as elevation.

⇝ chain complex

$$0 \longrightarrow \Delta^0(X,G) \xrightarrow{\delta} \Delta^1(X,G) \longrightarrow 0$$

$H^0(X,G) = \ker \delta$ = constant fns on each component

= direct product of components

(vs. direct sum, like in $H_0$ case)

$= \prod_{\substack{\text{components} \\ \text{of } X}} G.$

$H^1(X,G) = \Delta^1(X,G)/\operatorname{Im}\delta$.

For $f \in \Delta^1(X,G)$

$[f] = 0 \Leftrightarrow f \in \operatorname{Im}\delta \Leftrightarrow f$ is an antideriv.

$H^1(X,G) = \Delta^1(X,G)/\operatorname{Im}\delta$. $f \in \Delta^1(X,G)$

$[f] = 0 \Leftrightarrow f \in \operatorname{Im}\delta \Leftrightarrow f$ is an antideriv.

<u>Examples</u>. ① $X$ = tree

Antiderivs always exist.

$\Rightarrow H^1(X,G) = 0$.

② $X$ = 

$\Delta^1(X,G) \cong G$

No nonzero fn has antider.

$\Rightarrow H^1(X,G) \cong G$.

③ $X = \bigvee_\alpha S^1$

$H^1(X,G) = \prod_\alpha G$.

<u>More generally</u> $X$ = any (oriented) graph

$T$ = maximal tree/forest

$E$ = edges outside $T$.

$\rightsquigarrow H^1(X,G) = \prod_E G$.

exercise. Hint: first consider fns $\equiv 0$ on $T$...

Show any other $f \in \Delta^1$ is cohom. to such a fn.

## Two dimensions

X = 2-dim Δ-complex

$\delta : \Delta^1(X,G) \longrightarrow \Delta^2(X,G)$

$$\delta f([v_0,v_1,v_2]) = f(v_1v_2) - f(v_0v_2) + f(v_0v_1)$$

Check that

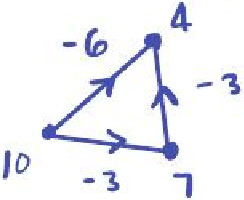
$$0 \longrightarrow \overset{f}{\Delta^0} \longrightarrow \overset{\delta f}{\Delta^1} \longrightarrow \overset{\delta\delta f}{\Delta^2} \to 0$$

is a chain complex:

$$\delta\delta f([v_0,v_1,v_2]) = (f(v_2)-f(v_1)) - (f(v_2)-f(v_0)) + (f(v_1)-f(v_0))$$

$$= 0.$$

If you hike/ski a loop, elevation change is 0.



What is a 1-cocycle? $(\ker \delta_1)$

$\delta f = 0 \Leftrightarrow f(v_0v_2) = f(v_0v_1) + f(v_1v_2)$

So: f is locally an antideriv.

↑ on any triangle.

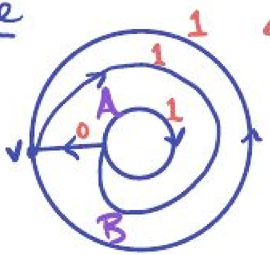
When is f a 1-coboundary? $(\operatorname{Im} \delta_0)$

When it's an antideriv.

So a nontrivial elt of $H^1(X)$ is a fn on edges that is locally, but not globally an antideriv.

## Example

$X =$



$f \in \Delta^1(X, \mathbb{Z})$

Check $f$ is a 1-cocycle

i.e. $\delta f = 0$.

$\delta f(A) = 0$

$\delta f(B) = 0$

But $f$ is not in $\operatorname{im} \delta_0$:

Any value of $f(v)$ fails.

Mar 16

# COHOMOLOGY

$\Delta^i(X)$ = functions from $i$-simplices of $X$ to $G$.

= homoms $\Delta_i(X) \to G$

$$\delta : \Delta^i(X,G) \to \Delta^{i+1}(X,G)$$
$$f \mapsto \delta f$$

for $f \in \Delta^i$, $\sigma = (i+1)$-simplex

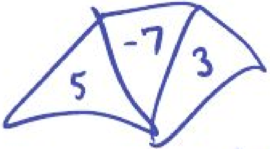
$$\delta f(\sigma) = \Sigma(-1)^k f(\partial_k \sigma) \quad ⊛ \leftarrow \text{same}$$

Claim: $\delta^2 = 0$. $\quad c = {}^{i+1}\text{chain} \quad f = {}^{i}\text{cochain}$

Check: $\delta f(c) = f(\partial c)$ ⊛

$$\delta\delta f(c) = \delta(f\partial c) = f\partial^2 c \qquad c = i+2\text{-chain}$$
$$= f(0) = 0$$

$i = 2$



Claim $\leftrightarrow$ $\Delta^i$ form a chain complex:

$\operatorname{im}\delta_{-1} \subseteq \ker \delta_i$

$H^*(X,G)$ = homology of this chain complex.

$$H^i(X,G) = \frac{\ker \delta_i}{\operatorname{im} \delta_{i-1}}$$

## Two dimensions

$X = 2\text{-dim } \Delta\text{-complex}$

$\delta : \Delta^1(X,G) \longrightarrow \Delta^2(X,G)$

$$\delta f([v_0,v_1,v_2]) = f(v_1v_2) - f(v_0v_2) + f(v_0v_1)$$

Check that

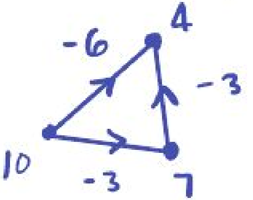
$$0 \longrightarrow \overset{f}{\Delta^0} \xrightarrow{\delta_0} \overset{\delta f}{\Delta^1} \xrightarrow{\delta_1} \overset{\delta\delta f}{\Delta^2} \rightarrow 0$$

is a chain complex:

$$\delta\delta f([v_0,v_1,v_2]) = (f(v_2)-f(v_1)) - (f(v_2)-f(v_0)) + (f(v_1)-f(v_0))$$

$$= 0.$$

If you hike/ski a loop, elevation change is 0.



What is a 1-cocycle? ($\ker \delta_1$)

$$\delta f = 0 \iff f(v_0v_2) = f(v_0v_1) + f(v_1v_2)$$

So: f is locally a deriv.

↑ on any triangle.

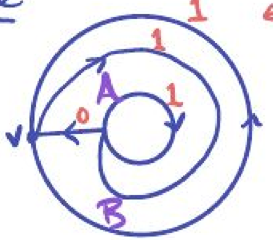
When is f a 1-coboundary? ($\operatorname{Im} \delta_0$)

When it's a deriv.

So a nontrivial elt of $H^1(X)$ is a fn on edges that is locally, but not globally an a deriv.

## Example

$X =$



$f \in \Delta^1(X, \mathbb{Z})$

Check $f$ is a 1-cocycle

i.e. $\delta f = 0$.

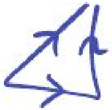
$\delta f(A) = 0$

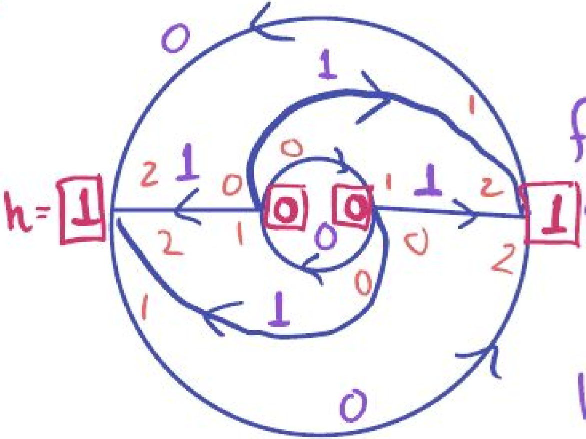
$\delta f(B) = 0$

But $f$ is not in $\operatorname{im} \delta_0$:

Any value of $f(v)$ fails.

## Easier example





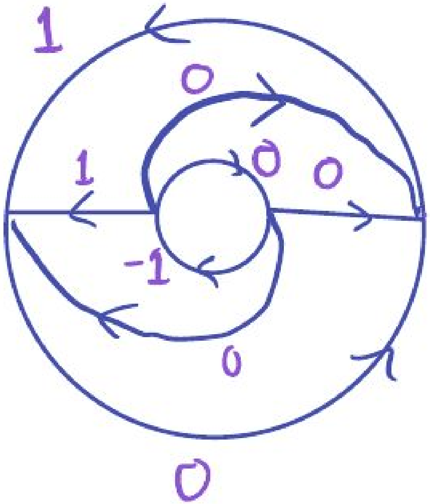
$f$ a 1-cocycle

It is a cobound.

It is $\delta h$

So: trivial elt of $H^1(X)$.

# Another try.

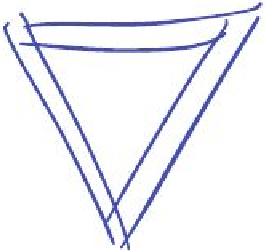


f a 1-cycle

It is not a cobound. because the outside (and inside loops are nonzero).

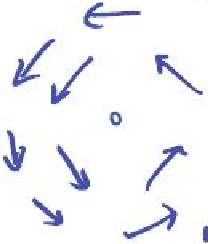
So: nontrivial elt of $H^1$.

# Think about



Google: triangle optical illusion.

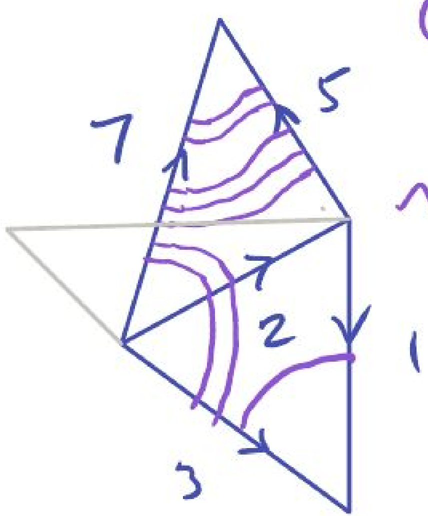
& DeRahm cohomology



This vect field on $\mathbb{R}^2 \setminus 0$ is closed: locally gradient

not exact: not globally a gradient

# Geometric interpretation of 1-cocycles.



Cocycle condition
$(2+5=7)$

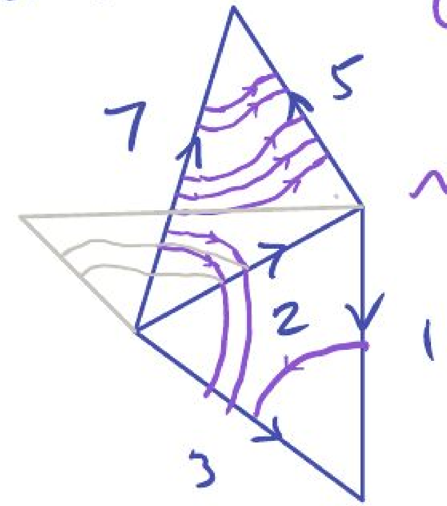
⇝ collection of arcs in each triangle

⇝ "curves" in $X$.

# Geometric interpretation of 1-cocycles.

Mar 18

dim $X = 2$

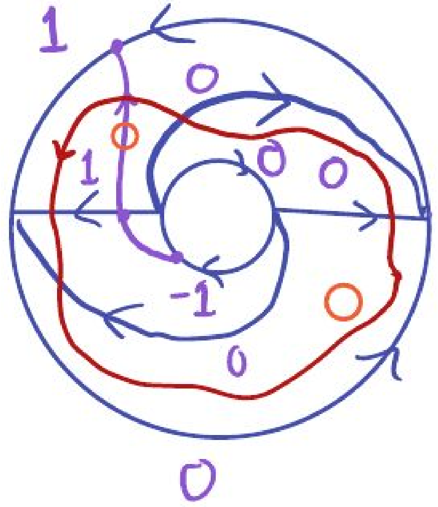


Cocycle condition

(2+5=7)

⇝ collection of arcs in each triangle

⇝ "curves" in $X$.



$n = 2$
$k = 1$

We'll see pairing

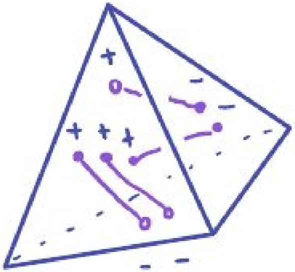
k-homology, (n-k)-cohomology

$( \bullet , \bullet ) = 1 \Longrightarrow$ both the hom & co cl. are $\neq 0$

The cohomology class is intersect with purple.

# The $n=3$ case

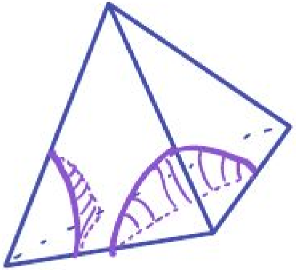
2-cocycles.



Cocycle condition: same # of incoming/outgoing dots.

The cohomology class is: intersect with purple.

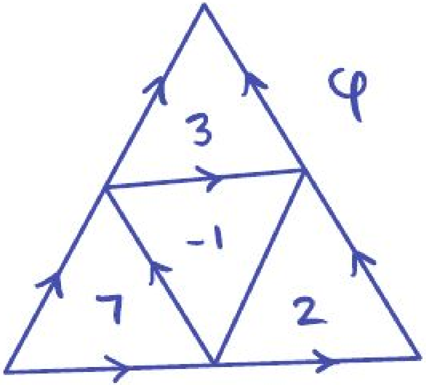
1-cocycles



This "eats" 1-chains.

# Examples of 2-cocycles

① $X = D^2$



$$\Delta^0 \longrightarrow \Delta^1 \xrightarrow{\delta_1} \Delta^2 \longrightarrow 0$$

Any elt of $\Delta^2$ is a cocycle.

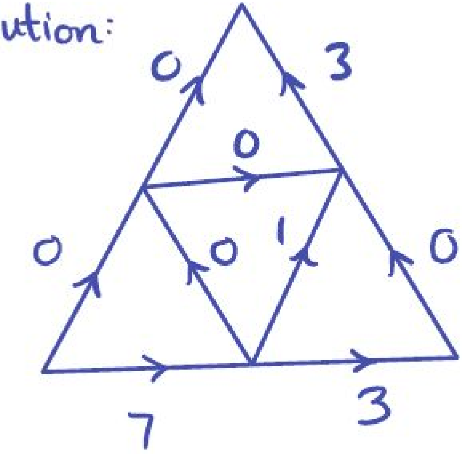
Is it trivial (in $\operatorname{Im} \delta_1$)

We know $H^2(D^2; \mathbb{Z}) = 0$.

So $\varphi = \delta\psi$  $\psi \in \Delta^1$

What is $\psi$?

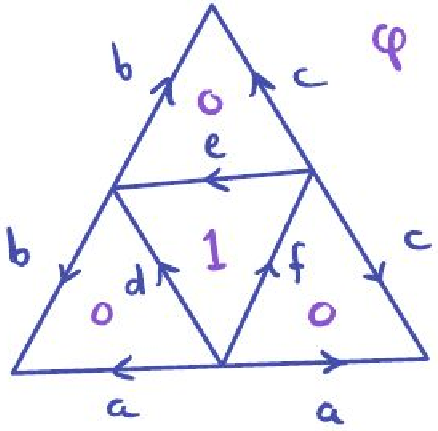
Solution:



antiderivative.

No obstruction.

② $X = S^2$

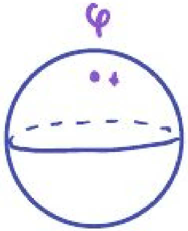


(Again every elt of $\Delta^2$ is a 2-cocycle since $\Delta^3 = 0$).

Want to show

$[\varphi] \neq 0$ in $H^2(S^2)$

i.e. no antideriv. $\psi$



Any $\psi$ can be thought of as 6 nums $a, \dots, f$

If we want $\delta\psi = \varphi$ then

Each $\Delta$ gives an equation

$$\left.\begin{array}{l} b+d=a \\ e+c=a \\ b+f=c \\ e+f=d+1 \end{array}\right\} \Rightarrow \underset{a}{\overset{\shortparallel}{(b+d)}} - \underset{a}{\overset{\shortparallel}{(e+c)}} = 1$$

contrad.

# How are homology & cohomology related?

We'll see (as abelian gps)

$$H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X)$$

where $T_n$ = torsion part of $H_n$.

for example. $X = \mathbb{R}P^2$

$H_0(X) = \mathbb{Z}$ $\quad H^0(X) = \mathbb{Z}$

$H_1(X) = \mathbb{Z}/2$ $\quad H^1(X) = 0$

$H_2(X) = 0.$ $\quad H^2(X) = \mathbb{Z}/2$

↑ find it!

We'll also see:

$$H^1(X) = \mathrm{Hom}(H_1(X), \mathbb{Z})$$

Note: $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0.$

# Example of a chain complex

$$C: \; 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

$$\rightsquigarrow \; H_0 = \mathbb{Z} \quad H_1 = \mathbb{Z}/2 \quad H_2 = 0 \quad H_3 = \mathbb{Z}$$

$$\begin{tikzcd}
C^*: \; 0 & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "0"'] & \mathbb{Z} \arrow[l, "2"'] & \mathbb{Z} \arrow[l, "0"'] & 0 \arrow[l] \\
& \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[u, equal] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[u, equal] & & &
\end{tikzcd}$$

verify this!

$$\rightsquigarrow \; H^0 = \mathbb{Z} \quad H^1 = 0 \quad H^2 = \mathbb{Z}/2 \quad H^3 = \mathbb{Z}$$

# CUP, CAP, AND POINCARÉ DUALITY

**Poincaré duality.** $H^k(X) \xrightarrow{\cong} H_{n-k}(X)$

$$\varphi \longmapsto [M] \frown \varphi$$

**Also.** Under this isomorphism, cup product corresponds to intersection: $\varphi \cup \psi \longmapsto \varphi^* \cap \psi^*$

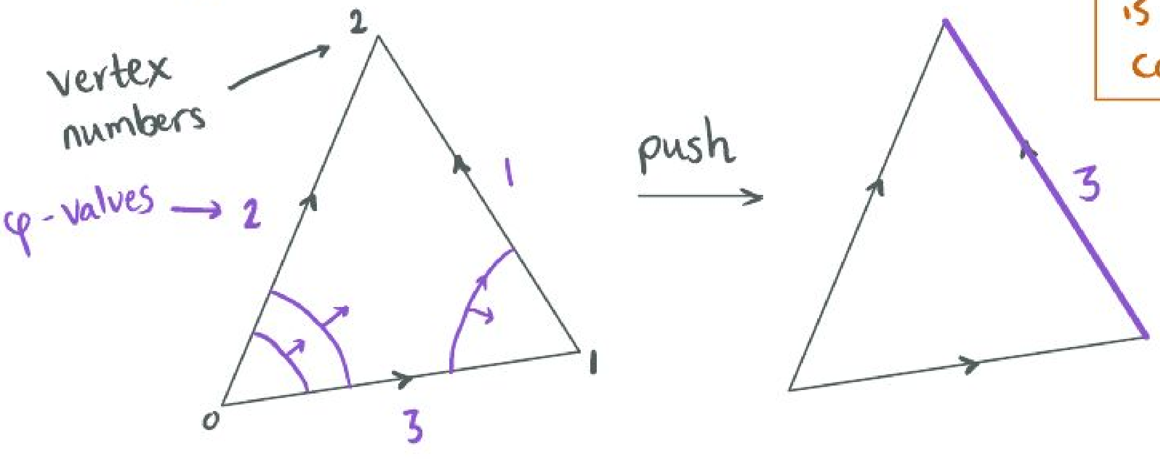
We'll work with $\Delta$-complexes, simplicial (co)homology.

## CAP

**Idea.** Realize cohomology class $\varphi$ as "intersect with dual object." Push dual in each simplex toward highest vertex (this is well-defined across different simplices in a $\Delta$-complex). Result is $[M] \frown \varphi = \varphi^*$

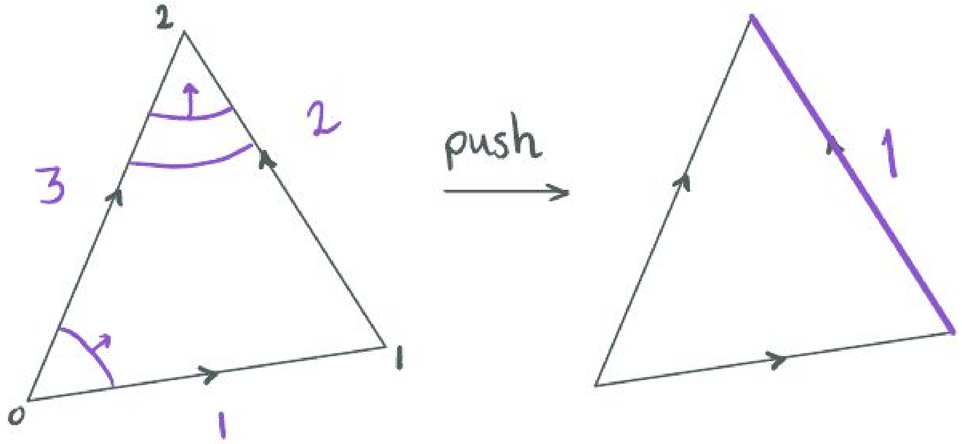
Note: in a manifold, orientation is same as co-orientation

**Example 1.** $n=2,\ k=1$

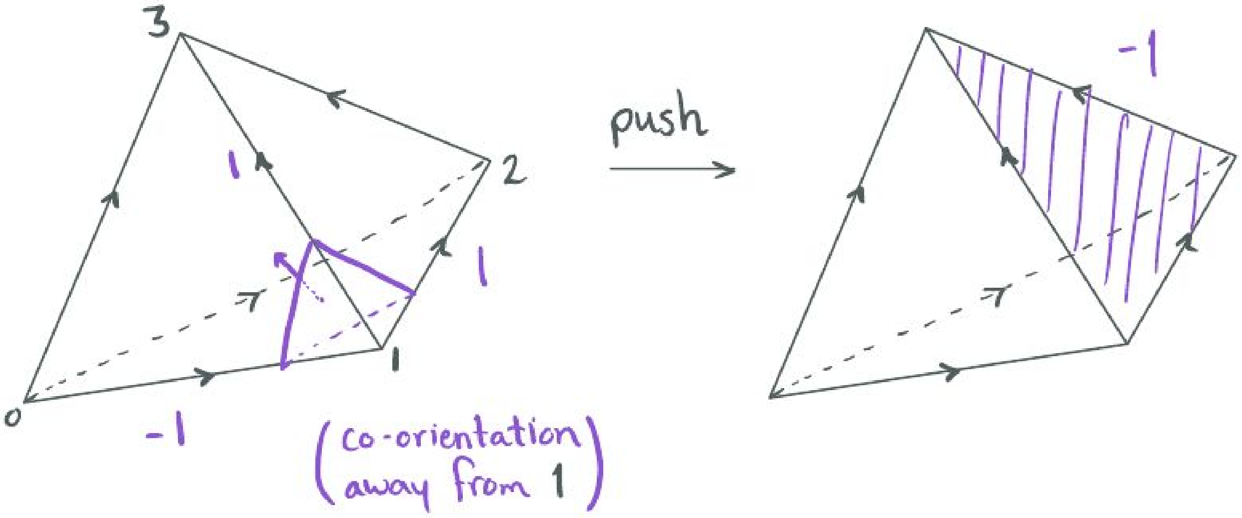


This is exactly what $\varphi \frown [M]$ gives!

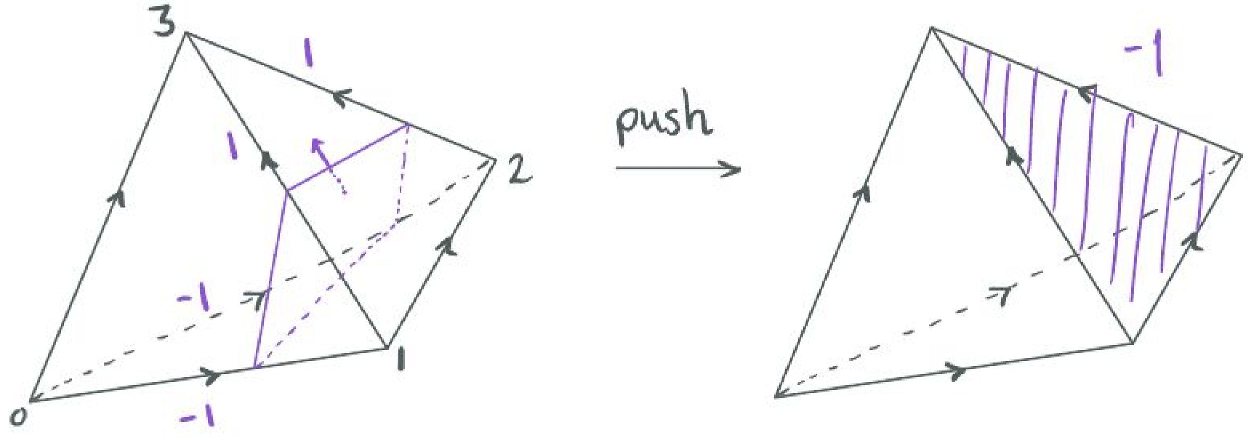
## Example 2. $n=2$, $k=1$



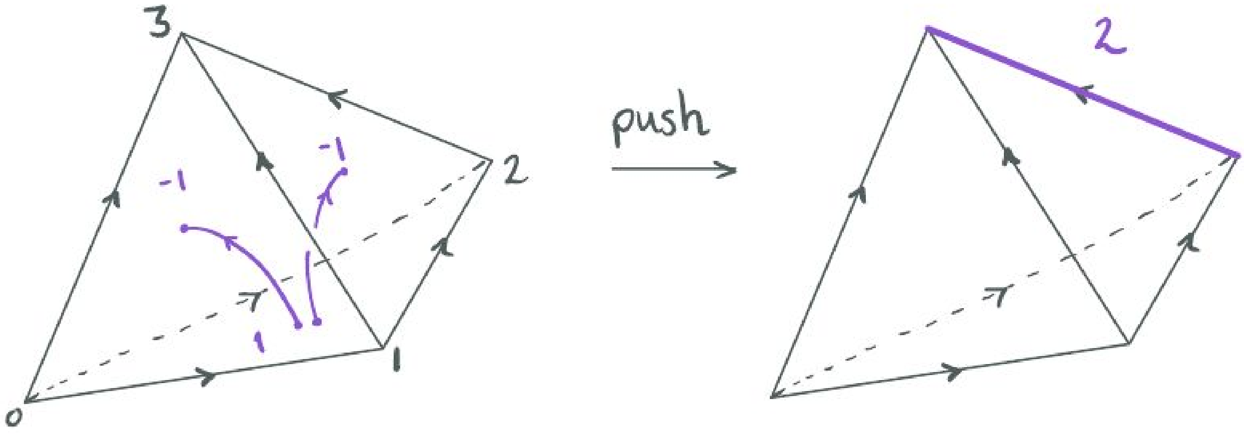
## Example 3 $n=3$, $k=1$



## Example 4 $n=3$, $k=1$



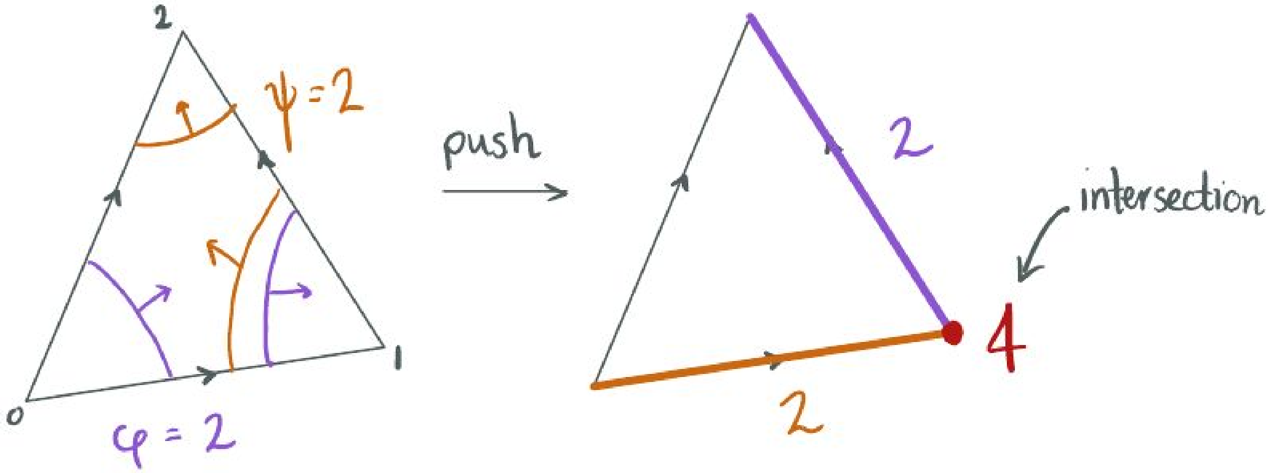
### Example 5 $n=3,\ k=2$



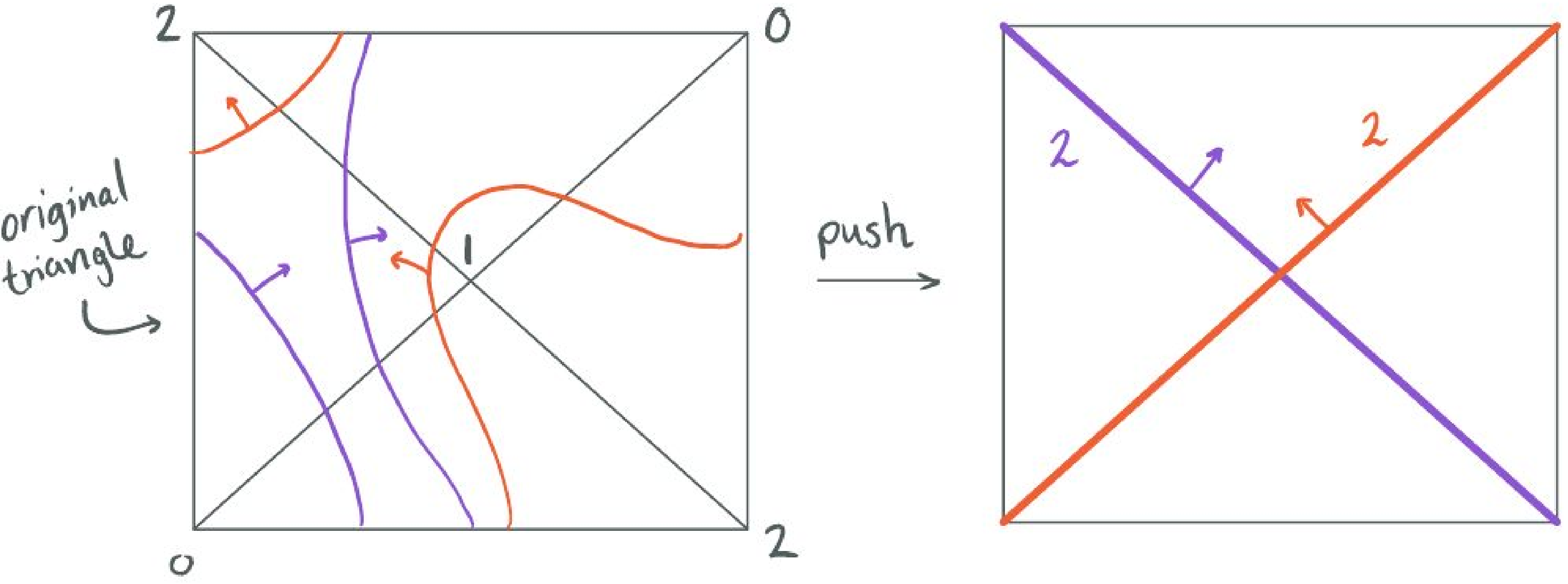
## Cup

Idea. To find $\varphi \cup \psi$, push $\varphi$ up, push $\psi$ down and intersect

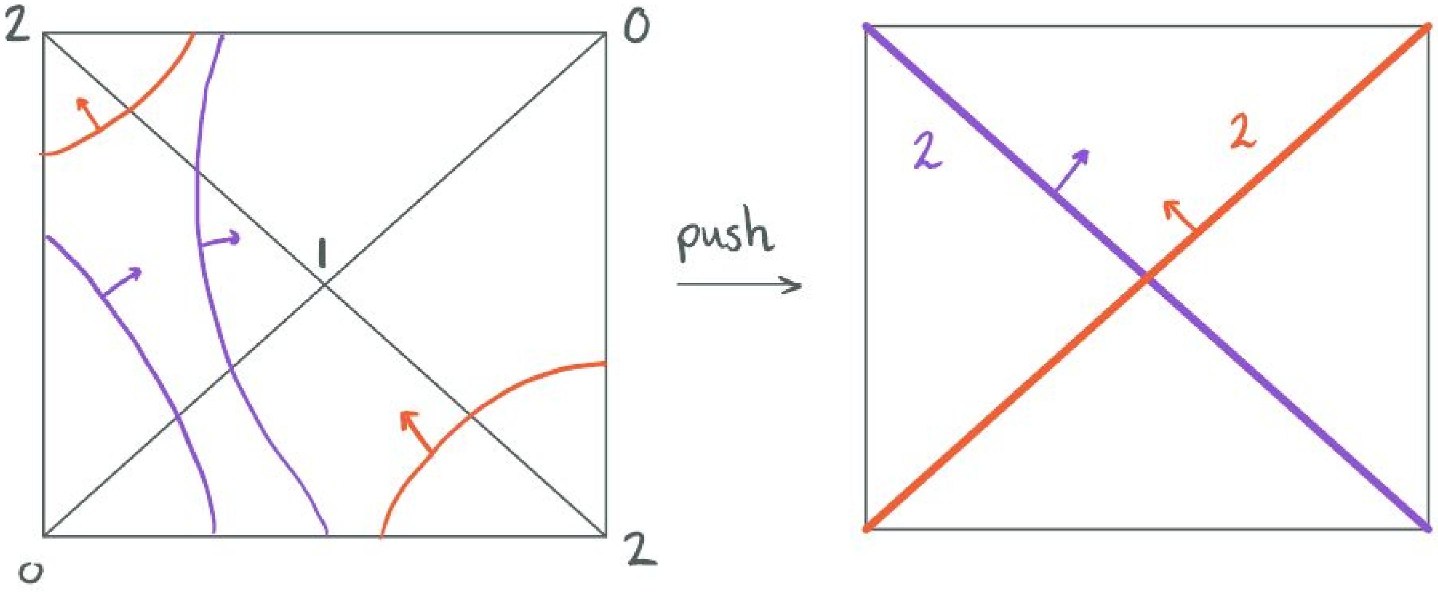
### Example 1. $n=2,\ k=1$



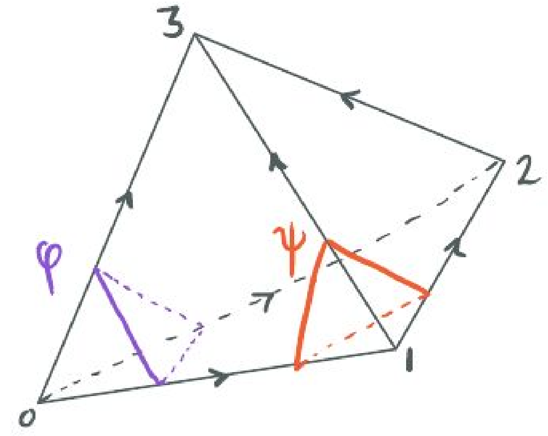
Can view same example in context of nearby triangles:



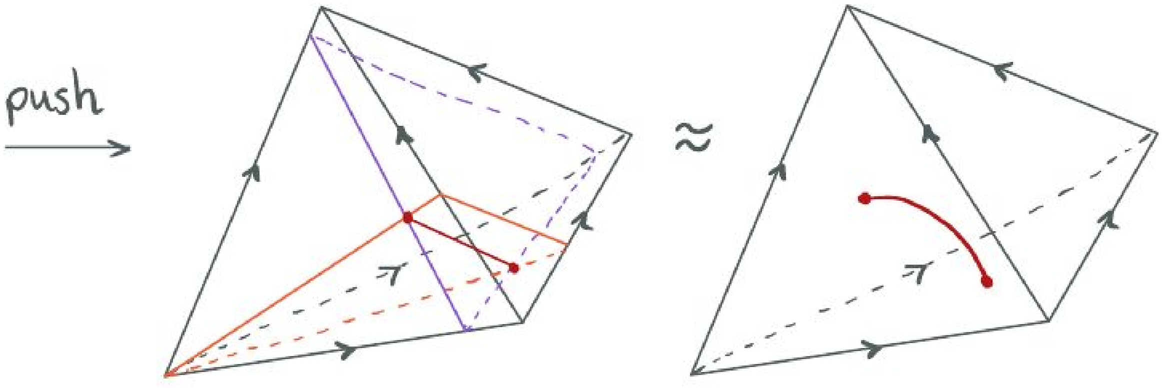
We can modify the curves by homotopy, giving cohomologous cochains:



## Example 2. $n=3$, $k=1,1$ (mod 2 this time)



Have $\varphi \cup \psi \in H^2 \rightsquigarrow$ should be dual to a 1-cell. If we push all the way and intersect, get a point (not what we want). If we push <u>almost</u> all the way, we get what we want:



This is dual to $\varphi \cup \psi$!

Note: In the earlier examples, pushing almost all the way also works.

Mar 30

Grad Student Top. Conf this weekend.

Why are we doing cohomology?

Product structures.

e.g. $\mathbb{C}P^\infty$ 1 cell in each even dim.

$H_i = \mathbb{Z}$ $\forall$ $i$ even.

$H^i(\mathbb{C}P^\infty) = \mathbb{Z}[\alpha]$ $\alpha \in H^2(\mathbb{C}P^\infty)$

$\alpha^k \in H^{2k}(\mathbb{C}P^\infty)$.

So all elts of $H^*$ are $\mathbb{Z}$-multiples of powers of a single elt $\alpha \in H^2(\mathbb{C}P^\infty)$.

Also: pairing b/w $H_*$ & $H^*$

Poincare duality...

# Cohomology theory

Reduced groups, relative cohomology, long ex seq of pairs, excision, Mayer-Vietoris all work for cohomology.

Induced homomorphisms — contravariant

Given $f: X \to Y$ get

$$f^\#: C^n(Y,G) \longrightarrow C^n(X,G)$$

$\varphi^{\in}$ $\qquad \varphi(\sigma) = \varphi(f(\sigma))$

$\uparrow$ n-simp in X

$\delta f^\# = f^\# \delta \Rightarrow f^\#$ pres. cocycle / cobound.

$\rightsquigarrow f^*: H^*(Y,G) \longrightarrow H^*(X,G)$.

cocycles to cocycles:

Say $\delta\varphi = 0$ $\quad \varphi \in C^n(Y,G)$

Want $\delta f^\# \varphi = 0$.

$\shortparallel$

$f^\# \delta\varphi$

Also: $(fg)^* = g^* f^*$, $id^* = id$

Say $X \longmapsto H^n(X,G)$ is a contravariant functor.

---

Homotopy invariance same as before

$$f \simeq g \Rightarrow f^* = g^*$$

In homology: $g_\# - f_\# = \partial P + P\partial$

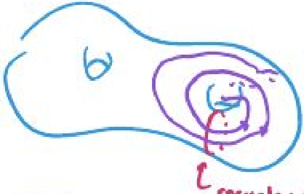
dualize: $g^\# - f^\# = \partial P^* + P^*\partial$.

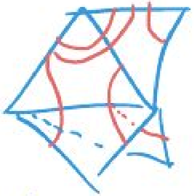
# Product Structures

## ① Evalvation pairing

$$H^k(X) \times H_k(X) \longrightarrow \mathbb{Z}$$

can use to show (co)cycles are nontrivial.





## ② Cup product

$$H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$

$$(\varphi, \psi) \longmapsto \varphi \cup \psi$$

$\rightsquigarrow H^*(X)$ is a graded ring.

e.g. polynomials

## ③ Cap product

$\dim X = n$

$$H^p(X) \times H_n(X) \longrightarrow H_{n-p}(X)$$

$$(\varphi, \alpha) \longmapsto \varphi \frown \alpha$$

Big goal:

**Poincaré Duality Thm**

$M$: compact, conn, oriented $n$-manifold

Then $$H^p(M) \longrightarrow H_{n-p}(M)$$

$$\varphi \longmapsto \varphi \frown [M]$$

is an isomorphism.

So all $p$-cocycles are: intersect with $n-p$ manifold.

i.e. all cocycles are nice.

# Cup Product

For $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$ R ring.
the cup product $\varphi \cup \psi \in C^{k+\ell}(X,R)$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

for $\sigma : \Delta^{k+\ell} \longrightarrow X$ a simplex.

We will see: cup product is intersection.

Need to show we get a product on level of cohomology: ① $\delta\varphi = \delta\psi = 0 \Rightarrow \delta(\varphi\cup\psi) = 0$

& ② $\varphi$ or $\psi$ cobound $\Rightarrow \varphi \cup \psi$ is.

So we'll get $H^k(X,R) \times H^\ell(X,R) \xrightarrow{\cup} H^{k+\ell}(X,R)$

**Lemma.** $\delta(\varphi\cup\psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

Lemma $\Rightarrow$ ① & ②.

**Pf of Lemma**

Say $\varphi \in C^k$, $\psi \in C^\ell$

$\sigma : \Delta^{k+\ell+1} \longrightarrow X$

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$(-1)^k \varphi \cup \delta\psi(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]})$$

Cup product is: assoc, distrib.

(since it is for cochains)

If R has 1 then $H^*(X,R)$ has 1

namely 1 in $H^0(X,R)$

Next time: Cup product on

orientable & nonorientable

surfaces.

Apr 1

# Cup products

$\varphi \in H^k(X)$

$\psi \in H^l(X)$

$\varphi \cup \psi \in H^{k+l}(X)$

$\varphi \cup \psi([v_0, \dots, v_{k+l}])$

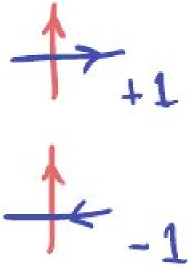
$= \varphi([v_0, \dots, v_k]) \cdot \psi([v_k, \dots, v_{k+l}])$

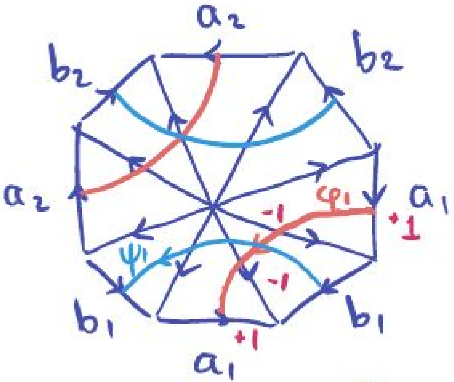
Example: Orientable Surfaces

$X = M_g$. We'll see:

$\cup \leftrightarrow \hat{i}$

cup algebraic int.





$M_2$

$a_i, b_i$ form basis for $H_1(M_g)$

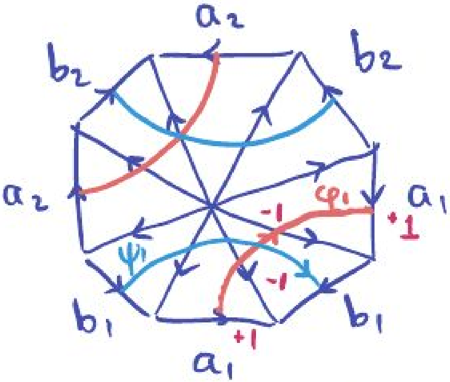
UCT $\Rightarrow H^1(M_g) \cong \text{Hom}(H_1(M_g), \mathbb{Z})$

Basis for $H_1 \rightsquigarrow$ basis for $H^1(M_g)$

$a_i \longleftrightarrow \varphi_i$

$b_i \longleftrightarrow \psi_i$

$$\varphi_i(a_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$



We know $H_2(M_g) = \mathbb{Z}$

UCT + $H_1$ free $\Longrightarrow H^2(M_g) \cong \mathrm{Hom}(H_2(M_g), \mathbb{Z}) \cong \mathbb{Z}$

Which elt of $H^2(M_g) \cong \mathbb{Z}$ is $\varphi_1 \cup \psi_1$?

By above : $(\varphi_1 \cup \psi_1)([M_g]) = 1$

↑ fund. class
all the Δ's with +1

$\varphi_1 \cup \psi_1 \in H^2(M_g)$

6 of the Δ's only see one of the two so $\varphi_1 \cup \psi = 0$.

Southern Δ : $0 \cdot 0 = 0$.

SE Δ : $+1 \cdot +1 = 1$

Notice: $\varphi_1, \psi_1$ intersect in SE triangle.

This tells us:

$\varphi_1 \cup \psi_1 \neq 0$ in $H^2$
$[M_g] \neq 0$ in $H_2$
& both generate $H^2$ & $H_2$

Nonorientable surfaces: See book or 2012 notes.

## Naturality

$\alpha, \beta \in H^*(Y)$

$f : X \to Y$

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

already true on cochain level.

Also: cup product for relative cohomology.

## The Cohomology Ring

Define $H^*(X;R) = \bigoplus_k H^k(X;R)$

Elts are finite sums.

$\smile$ induces ring structure.

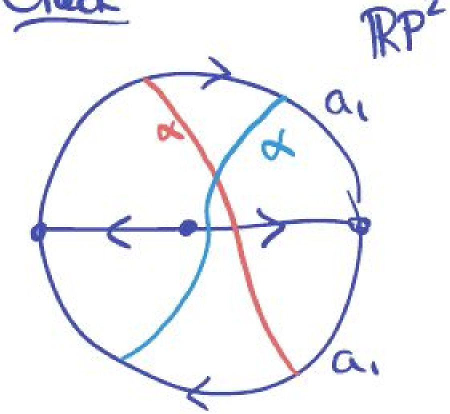
write $xy$ for $x \smile y$.

Claim: $H^*(\mathbb{RP}^2; \mathbb{Z}/2) = \{a_0 + a_1\alpha + a_2\alpha^2 : a_i \in \mathbb{Z}/2\}$

$= \mathbb{Z}/2[\alpha] / \langle \alpha^3 \rangle$

$H^1(\mathbb{RP}^2; \mathbb{Z}) = \langle \alpha \rangle$.

## Check



$H^0(\mathbb{RP}^2; \mathbb{Z}/2) = \mathbb{Z}/2 = 1$

$H^1(\mathbb{RP}^2; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha \rangle$

$H^2(\mathbb{RP}^2; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha^2 \rangle$

↑ proven below.

What is $\alpha \smile \alpha$?

$\alpha \smile \alpha \ (\text{top } \Delta) = 0 \cdot 1 = 0$

$\alpha \smile \alpha \ (\text{bot } \Delta) = 1 \cdot 1 = 1$

Check:

$(\alpha \smile \alpha)([\mathbb{RP}^2]) = 1$

$\Rightarrow$ $\alpha \smile \alpha = \alpha^2$ is the generator for $H^2(\mathbb{RP}^2)$

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# THE COHOMOLOGY RING

Last time:

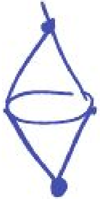
$$H^*(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^3)$$

$\alpha$ is the nonzero elt of $H^1(\mathbb{RP}^2;\mathbb{Z}/2)$

The deg of a ~~poly.~~ monomial. tells you the deg of the corr. elt. of $H^*$.

"graded ring" $R = \bigoplus_{d} R_d$

$$R_p \times R_q \subset R_{p+q}$$



One can also show:

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1}) \qquad |\alpha| = 1$$

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]$$

$$H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha] \qquad |\alpha| = 2$$

---

There are spaces with same $H_k$ & $H^k$ gps $\forall k$ but different $H^*$ rings:

$S^1 \vee S^1 \vee S^2$ $\qquad$ $T^2$

---

There are distinct spaces with identical $H^*$:

$$H^*(S^3 \vee S^5) \cong H^*(S(\mathbb{CP}^2)) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(5)}$$

degrees.

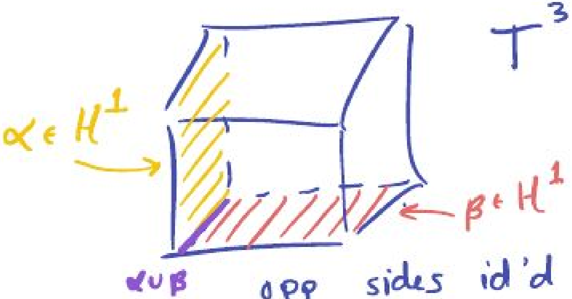
**A fact:** $\alpha \in H^k$, $\beta \in H^\ell$

**Prop.** $\alpha \cup \beta = (-1)^{k+\ell}(\beta \cup \alpha)$

if $R$ commutative.

**Next goal.** $H^*(T^n)$

$T^n = S^1 \times \cdots \times S^1$



We'll show:

$H^k(T^n;\mathbb{Z})$ = free abel. gp with basis

$$\alpha_{i_1} \cup \cdots \cup \alpha_{i_k} \quad i_1 < \cdots < i_k$$

where $\alpha_i \in H^1(T^n;\mathbb{Z})$ is $p_i^*(\alpha)$ for $\alpha$ a gen. for $H^1(S^1;\mathbb{Z})$ & $p_i$ is proj. to $i^{\text{th}}$ factor.

To prove this we'll do something more general.

Kunneth formula for $H^*(X \times Y)$.

## Tensor products

$A, B$ abelian gps

$A \otimes B$ is the abel. gp gen by

$$a \otimes b \qquad a \in A,\ b \in B.$$

e.g. $5a_1 \otimes b_1 + 7a_2 \otimes b_2$

relations

$$(a + a') \otimes b = a \otimes b + a' \otimes b$$

$$a \otimes (b + b') = \text{similar.}$$

Bilinear maps $A \times B \longrightarrow C$ same as

homoms $A \otimes B \longrightarrow C$

## Cross product (or, external cup product):

$$H^*(X;\mathbb{Z}) \times H^*(Y;\mathbb{Z}) \to H^*(X \times Y;\mathbb{Z})$$

$$(a, b) \longmapsto p_1^*(\alpha) \cup p_2^*(\beta)$$

bilinear so have homom.

$$H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \to H^*(X \times Y;\mathbb{Z})$$

Multiplication on LHS:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} \ ac \otimes bd$$

**Thm** (Künneth formula). This is an isomorphism *if* $H^*(X;\mathbb{Z})$ or $H^*(Y;\mathbb{Z})$ are fin. gen & free in each degree.

## Exterior algebras

$\Lambda[\alpha_1, \alpha_2, \ldots, \alpha_n]$

As a gp: gen by

$\alpha_{i_1} \cdots \alpha_{i_k}, \quad i_1 < \cdots < i_k$

Multiplication: $\alpha_i \alpha_j = -\alpha_j \alpha_i$.

$A, B$ gps

$A \wedge B$ gen by $a \wedge b$ with all $\otimes$ relations *and* $a \wedge b = -b \wedge a$.

in particular: $\alpha_i^2 = 0$.

**Cor** $H^*(T^n;\mathbb{Z}) \cong \Lambda[\alpha_1, \ldots, \alpha_n]$

$|\alpha_i| = 1$

---

Example of bilinear map: dot product

$(5v + 7w) \cdot u = 5v \cdot u + 7w \cdot u$

$f : \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\text{dot}} \mathbb{R}$

$f(u+v, w) = f(u,w) + f(v,w)$

$\rightsquigarrow \mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}$

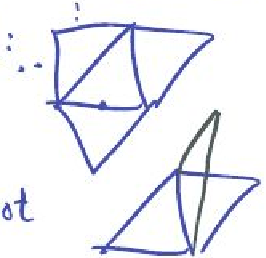
# POINCARÉ DUALITY

For $M$ a compact orientable $n$-manifold

$$H_k(M) \cong H^{n-k}(M)$$

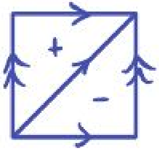
compact: finitely many $n$-simplices. $\sigma_1, \dots, \sigma_N$

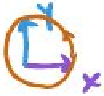
manifold:



not

e.g.





APR 6

Or by UCT, modulo torsion we have

$$H_k(M) \cong H_{n-k}(M)$$

## Examples

① $H_*(S^n)$ $\mathbb{Z}, 0, \dots, 0, \mathbb{Z}$

② $H_*(M_g)$ $\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$

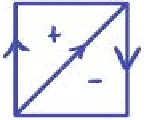
③ $H_*(T^n)$ $\mathbb{Z}^{\binom{n}{0}}$ $\mathbb{Z}^{\binom{n}{1}}$ $\dots$ $\mathbb{Z}^{\binom{n}{n-1}}$ $\mathbb{Z}^{\binom{n}{n}}$

The statement of PD gives an explicit $\cong$.

---

Orientable: $\exists\ \varepsilon_1, \dots, \varepsilon_N \in \{\pm 1\}$ s.t. $\sum_{i=1}^{N} \varepsilon_i \sigma_i$ is a cycle

" $[M]$ "



# The idea of PD

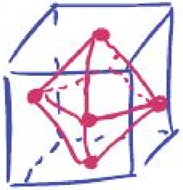
For manifolds:

cell structures ⟷ dual cell structures.

k-cells ⟷ (n-k)-cells

face relations reversed.

examples

① 

dual cell structures on $S^2$.

& other platonic solids

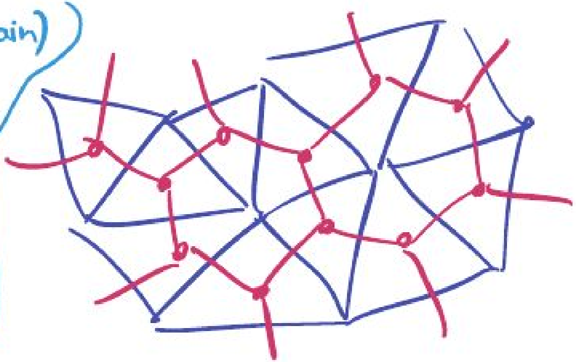
A pink object (chain) is a cocycle if when pair with any blue $\partial$, get 0

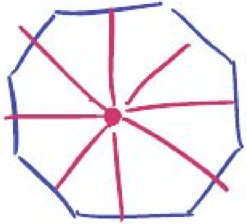
⇕

pink cycles

⇓

PD.

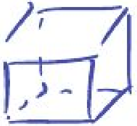


② 

$M_g$

dual cell str. ≅ orig. cell str.

③ $T^n$



ditto

# Duality with $\mathbb{Z}/2$ coeffs

Can ignore signs. $\rightsquigarrow$ natural pairing between cell str $C$ & dual $C^*$

$$C_i \leftrightarrow C^*_{n-i}$$

Under this identification

$$\partial : C_i \longrightarrow C_{i-1}$$

$$\sigma \longmapsto \text{sum of faces.}$$

becomes

$$\delta : C^*_{n-i} \longrightarrow C^*_{n-i+1}$$

$$\sigma^* \longmapsto \text{sum of dual cells of which } \sigma^* \text{ is a face.}$$

$$\rightsquigarrow H_i(C;\mathbb{Z}/2) \cong H^{n-i}(C^*;\mathbb{Z}/2).$$

$$\begin{matrix} H_i(C;\mathbb{Z}/2) & \cong & H^{n-i}(C^*;\mathbb{Z}/2) \\ \cong & & \cong \\ H_i(M;\mathbb{Z}/2) & & H^{n-i}(M;\mathbb{Z}/2) \end{matrix}$$

This proves PD for $\mathbb{Z}/2$ coeffs.

## Cap product

$k \geq \ell$

$$\frown : C_k(X) \times C^\ell(X;\mathbb{Z}) \longrightarrow C_{k-\ell}(X)$$

$$(\sigma, \varphi) \longmapsto \underbrace{\varphi(\sigma|_{[v_0,\dots,v_\ell]})}_{\text{number}}\, \sigma|_{[v_\ell,\dots,v_k]}$$

As usual, need to check this induces a map on co/homology. The required formula is:

$$\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

$\rightsquigarrow$ cycle $\frown$ cocycle = cycle

cycle $\frown$ coboundary = boundary

boundary $\frown$ cocycle = boundary

Two facts:

- linear in each var
- natural

$f : X \to Y$

$$f_*(\sigma) \frown \varphi = f_*(\sigma \frown f^*(\varphi))$$

in $H_*(Y)$

## Thm (PD)

$M$ = compact $n$-manifold with orientation $[M]$. Then

$$H^k(M) \longrightarrow H_{n-k}(M)$$
$$\varphi \longmapsto [M] \frown \varphi$$

is an $\cong$.

Apr 8

Poincaré Duality $M$ compact, orientable $n$-manifold.

$$H^{n-k}(M) \longrightarrow H_k(M)$$
$$\varphi \longmapsto \varphi \frown [M]$$

makes sense with $\mathbb{Z}$-coeff exactly when $M$ is orientable.

Orientable: can put $\pm 1$ on each simplex of $M$ and get an $n$-cycle.

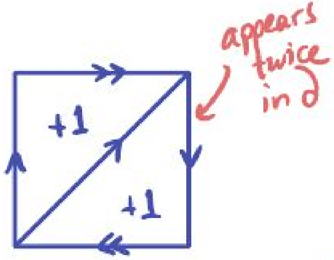
Note: every manifold is $\mathbb{Z}/2$-orientable. If you put (+)1 on each $n$-simplex, get a $\mathbb{Z}/2$ cycle.

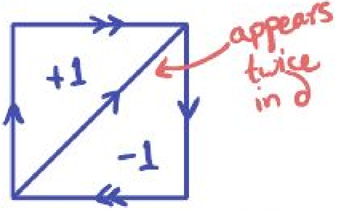
$\overset{n-k \text{ cocycle}}{\varphi} \frown \overset{n \text{ cycle}}{[M]}$ is a $k$-cycle.

on each simplex $\sigma$ of $M$, evaluate $\varphi$ on "front" $n-k$ simplex of $\sigma$

$\rightsquigarrow$ number

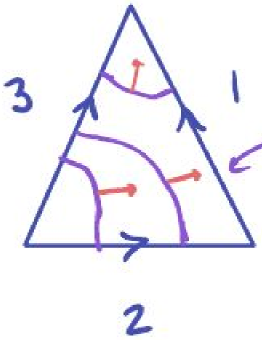
Scale the "back" $k$ simplex of $\sigma$ by that number.





$\Rightarrow$ not orientable

We know: cocycles $\rightsquigarrow$ dual objects



level curves.

they are co-oriented.

Recall: $\varphi \in H^k$

$\psi \in H^\ell$

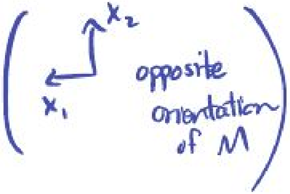
$\varphi \smile \psi \in H^{k+\ell}$

Two claims: ① Cup product of cocycles is intersection of dual objects.

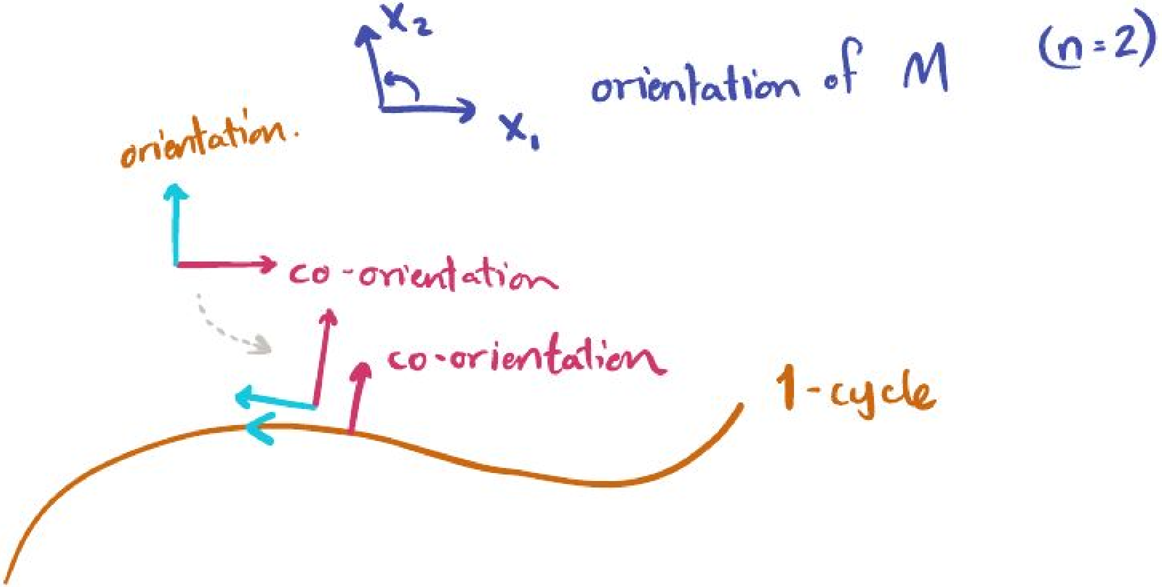
② Cap product is "pushing" or homotoping the dual objects

In an orientable manifold :

co-orientations ⟷ orientations



opposite orientation of M



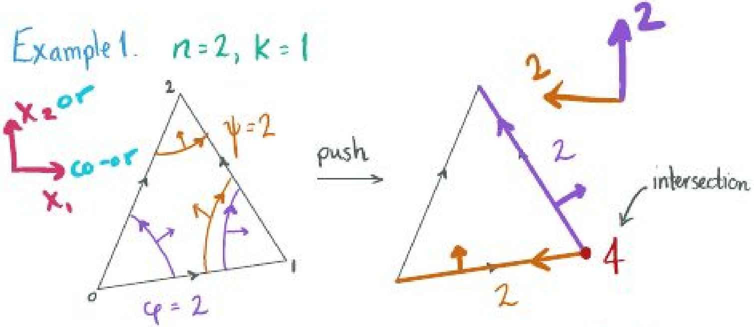
# Cup & Cap notes from my Teaching page

Apr 8

## Cup

Idea. To find $\varphi \cup \psi$, push dual of $\varphi$ up, push dual of $\psi$ down and intersect

Example 1. $n=2$, $k=1$



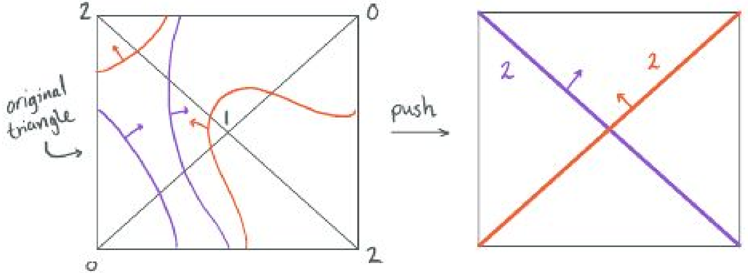
$\varphi, \psi \in H^1$

$\varphi \cup \psi \in H^2$

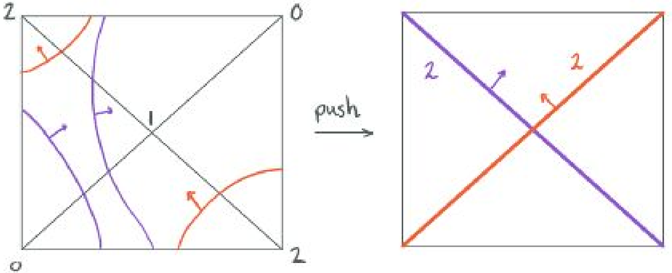
Using def of $\cup$:

$$\varphi \cup \psi(\sigma) = 2 \cdot 2$$
$$= 4$$

Can view same example in context of nearby triangles:



We can modify the curves by homotopy, giving cohomologous cochains:



# CUP, CAP, AND POINCARÉ DUALITY

Poincaré duality. $H^k(X) \xrightarrow{\cong} H_{n-k}(X)$

$\varphi \longmapsto [M] \cap \varphi$

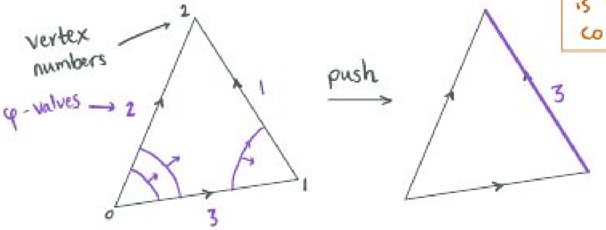
Also. Under this isomorphism, cup product corresponds to intersection: $\varphi \cup \psi \longmapsto \varphi^* \cap \psi^*$

We'll work with $\Delta$-complexes, simplicial (co)homology.

## CAP

Idea. Realize cohomology class $\varphi$ as "intersect with dual object." Push dual in each simplex toward highest vertex (this is well-defined across different simplices in a $\Delta$-complex). Result is $[M] \cap \varphi = \varphi^*$

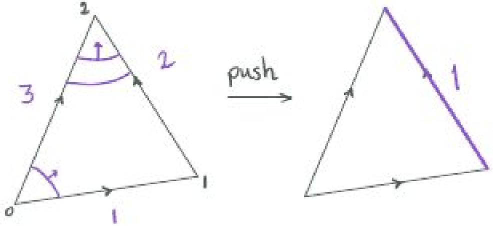
Example 1. $n=2,\ k=1$



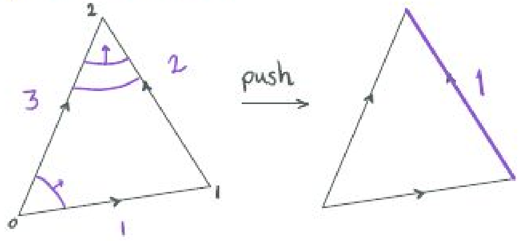
Note: in a manifold, orientation is same as co-orientation

This is exactly what $\varphi \cap [M]$ gives!

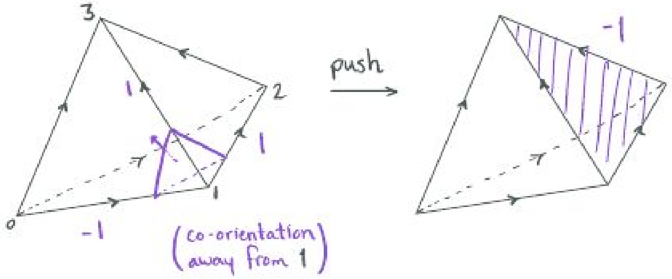
Example 2. $n=2,\ k=1$



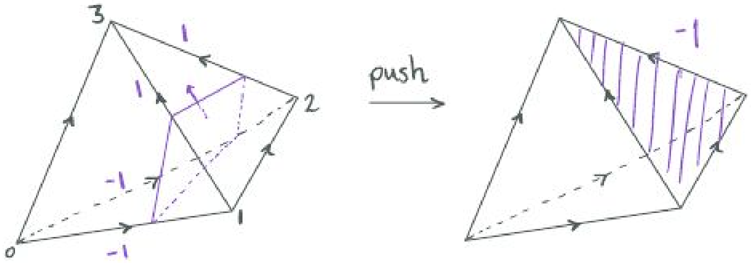
## Example 2 $n=2$, $k=1$



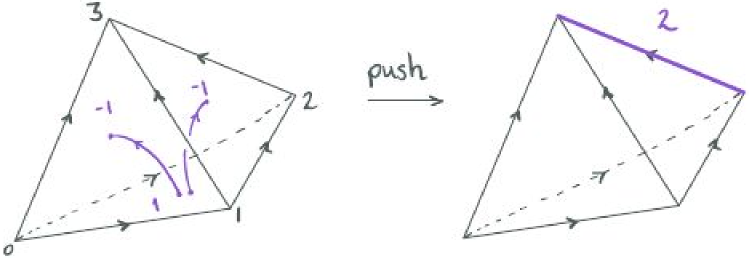
## Example 3 $n=3$, $k=1$



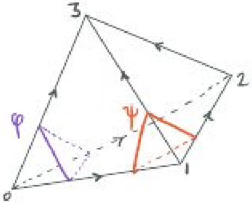
## Example 4 $n=3$, $k=1$



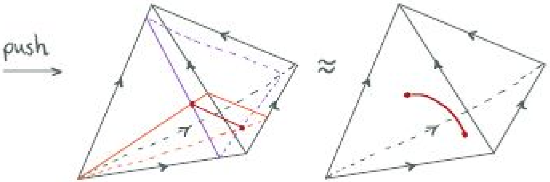
## Example 5 $n=3$, $k=2$



Example 2. $n=3$, $k=1,1$ (mod 2 this time)



Have $\varphi \cup \psi \in H^2 \rightsquigarrow$ should be dual to a 1-cell. If we push all the way and intersect, get a point (not what we want). If we push almost all the way, we get what we want:



This is dual to $\varphi \cup \psi$!

Note: In the earlier examples, pushing almost all the way also works.

Claim. This proves P.D.

# Applications of PD

PD: modulo torsion, have.

$$\begin{tikzcd} H^k(M) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong\ \text{UCT}"] & H_{n-k}(M) \\ H_k(M) \end{tikzcd}$$

① Euler characteristic.

**Prop.** dim $M$ odd $\Rightarrow$ $\chi(M) = 0$.

**Pf.** $\chi(M) \cong \sum_{k=0}^{n} (-1)^k \operatorname{rk} H_k(M;\mathbb{Z})$

Apply PD $\square$

APR 11

e.g. $\mathbb{Z} \quad 0 \quad 0 \quad \mathbb{Z}$

rk: $1 \quad 0 \quad 0 \quad 1$

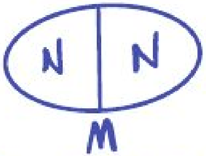
$\rightsquigarrow +1 - 0 + 0 - 1 = 0$.

**Prop.** dim $M$ even. $\chi(M)$ odd $\Longrightarrow$ $M$ is not a boundary of $(n+1)$-man.

e.g. $M = \mathbb{RP}^2$.

**Pf.** Suppose $M = \partial N$

$2N$:



$\overset{\overset{0}{\shortparallel}}{\chi(2N)} = 2\chi(N) - \chi(M)$. cont. $\square$

## ② Intersection Forms

$F$ = field. e.g. $\mathbb{Q}$.

$H_k(M;\mathbb{Q}) = H_k(M;\mathbb{Z})/\text{torsion} \otimes \mathbb{Q}$.

Have:

or: intersect

$$\langle\ ,\ \rangle : H_k(M) \times H_{n-k}(M) \xrightarrow{\cup^*} F$$

"intersection form"

Prop. Int. form is nonsingular.

i.e. $\forall\ \alpha \neq 0$, $\langle \alpha, \cdot \rangle \not\equiv 0$.

i.e. $\alpha \neq 0$ in $\mathrm{Hom}(H_{n-k}, F)$.

Pf. $H_k(M) \xrightarrow{PD} H^{n-k}(M) \xrightarrow{UCT} \mathrm{Hom}(H_{n-k}, F)$

$\xrightarrow{PD} \mathrm{Hom}(H_k, F)$.

These are all $\cong$ □

Now say $\dim M = n = 2k$ even.

$\rightsquigarrow$ int. form on $H_k(M)$.

$\rightsquigarrow$ $H_k(M) \times H_k(M) \longrightarrow \mathbb{Z}$

$\uparrow$ tors. free part of $\mathbb{Z}$ homol.

$\rightsquigarrow$ $\mathbb{Z}$-matrix $\langle \alpha_i, \alpha_j \rangle$

Prop. This matrix/int. form is unimodular, i.e. det 1.

Pf. Let $\alpha_1, \dots, \alpha_\ell$ basis.

$\beta_1, \dots, \beta_\ell$ dual basis (by prev. Prop)

$\rightsquigarrow$ $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$

Change of basis has det 1 □

If $n = 4k$, int. pairing is symmetric.

$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$$

$\Rightarrow$ eigenvals real.

$b_{2k}^{+}, b_{2k}^{-}$ # of pos, neg eigenvals.

$$\sigma(M) = b_{2k}^{+} - b_{2k}^{-}$$

"signature"

Let $\Omega_d^{SO}$ = cobordism group of d-dim. compact oriented manifolds

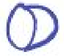
$M \sim N$ if

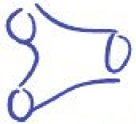
$M - N = \partial W$

addition: disjoint union.

id: $\emptyset$.

$\Omega_1^{SO} = 0$





**Thm.** $\Omega_d^{SO} = 0$ $d \leq 3$ (easyish for $d \leq 2$)

**Thm (Thom).** In dim 4, $\sigma$ is a cobordism invt. $\Rightarrow$ $\Omega_4^{SO}$ has $\mathbb{Z}$-many elts.

$\rightsquigarrow$ Fields medal

# Alexander duality

**Thm.** K compact, locally compact, nonempty subspace of $S^n$

$$\Rightarrow \tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$$

$\forall i$.

Surprisingly: LHS does not depend on the embedding.

**application.** $H_1(S^3 \setminus \text{knot}) \cong \mathbb{Z}$.

Gordon-Luecke: $\pi_1(S^3 \setminus \text{knot})$ determines the knot.

**Cor.** $X \subseteq \mathbb{R}^n$ compact, loc comp.

$\Rightarrow H_i(X; \mathbb{Z}) = 0 \quad i \geq n$

= torsion free $\quad i = n-1, n-2$.

**Pf.** $\tilde{H}^{-k} = 0$

$\tilde{H}^0, \tilde{H}^1$ always torsion free

**Application.** Klein bottle does not embed in $\mathbb{R}^3$ or $S^3$.

$H^2(K.B.)$ has torsion.

# SPECTRAL SEQUENCES

The LES for a pair gives $H_*(X)$ in terms

$H_*(X,A)$, $H_*(A)$

Similar LES for triple.

Not for quadruples.

Answer: Spectral seqs.

Apr 13

Filtration

$X$ = CW complex

Filter by subcomplexes

$X_0 \subseteq X_1 \subseteq \cdots$

$\rightsquigarrow$ filtration of $C_*(X)$: $\overset{\substack{\text{k-chains} \\ \text{in } X_p \\ \parallel}}{F_p C_k}$

$\rightsquigarrow$ assoc. graded modules:

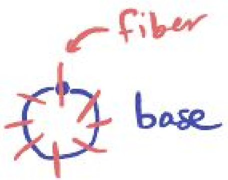
$$G_p C_k = F_p C_k / F_{p-1}(C_k)$$

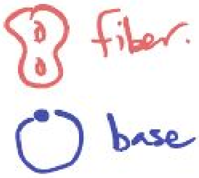
"new k-chains at stage p"

<u>Examples</u> ① $X_i = X^{(i)}$ i-skeleton.

② For a fiber bundle

$X_i$ = preimage of i-skeleton of base.





e.g. product.

or: Möbius band.

Other examples of fiber bundles:

SO(n)

$$\begin{tikzcd} SO_2 \arrow[r] & SO_3 \arrow[d] \\ & S^2 \end{tikzcd}$$

image of north pole.

⇒ can understand $H_* SO(n)$ by induction & Spec Seq.

Recall: $G_p C_k = F_p C_k / F_{p-1}(C_k)$

## Filtered Chain complexes

We have $\partial F_p C_k \subseteq F_p C_{k-1}$

⇝ induced $\partial : G_p C_k \longrightarrow G_p C_{k-1}$

⇝ associated graded chain complex

$$(G_p C_*, \partial)$$

Hope. $H_*(G_p C_*)$ easy-ish to compute, understand $H_*(X)$ as a limit...

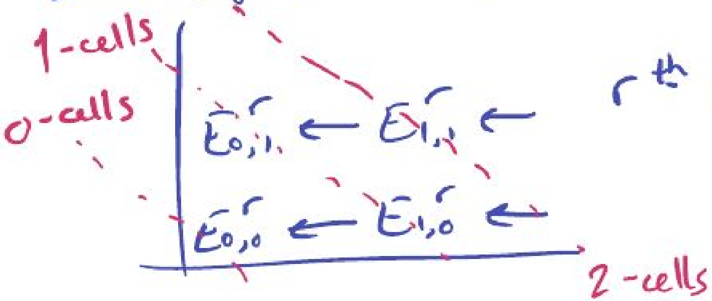
# Another idea of what a spectral seq is

A spectral seq has pages.

Each page is a 2D grid of vect spaces. (first quadrant)
(if we work over a field)

There are $\underset{\text{(maps)}}{\text{differentials}}$, and we get from one page to the next by taking $\underset{\substack{\parallel \\ \text{ker/im.}}}{\text{homology}}$.

Each page looks like



$r^{th}$ page

Taking homology turns the page

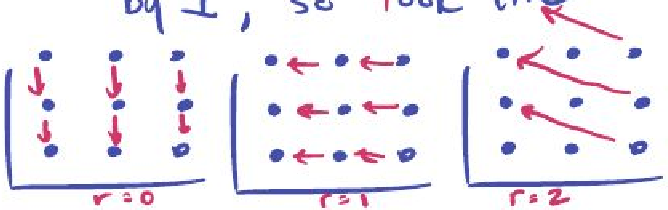
Arrows/maps change direction

Keep doing this forever.

The $E^r_{p,q}$ will correspond to k-chains where $p+q=k$.

e.g. $E^0_{p,q} = G_pC_{p+q}$

The differentials reduce the dim by 1, so look like



In favorable cases, each $E^r_{p,q}$ stabilizes with $r$.

$\rightsquigarrow E^\infty_{p,q}$ is this term.

$H_k(X)$ given by $E^\infty_{p,q}$
with $p+q=k$.

Think about paintball:
- Basis elts of the $E_{p,q}$ are players.
- All get one paintball.
- If you shoot a nontrivial player, both you & the target are eliminated.
- If you shoot trivial player, get to stay for next rand.

Sometimes a spectral seq degenerates, meaning all terms stabilize at the same time.

## Using spectral sequences

In LES, hope for 0's.

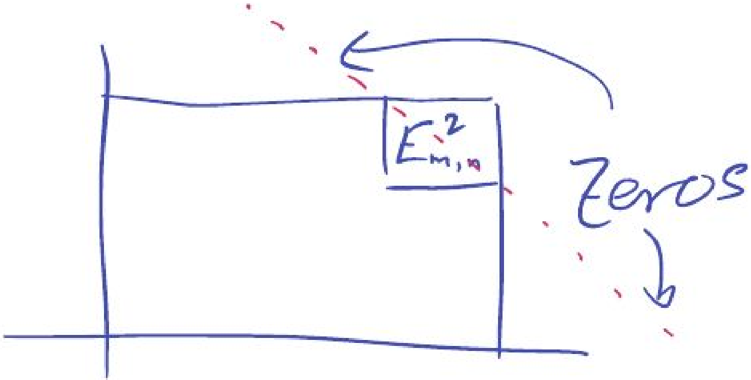
Same here.

For example, in Serre SS for fiber bundles

$$E^2_{p,q} = H_p(B, H_q(F))$$

If $B$ is $m$-dim & $F$ is $n$ dim.

then.



$$\Longrightarrow E^2_{m,n} = E^\infty_{m,n} = H_{m+n}(X)$$

this is a good excuse for the indexing.

# Spectral Sequences

## Cartoon

Apr 15

$X_0 \subseteq X_1 \subseteq \cdots$

$\bigcup X_i = X$ CW complex

$F_p C_k$ k-chains in $X_p$

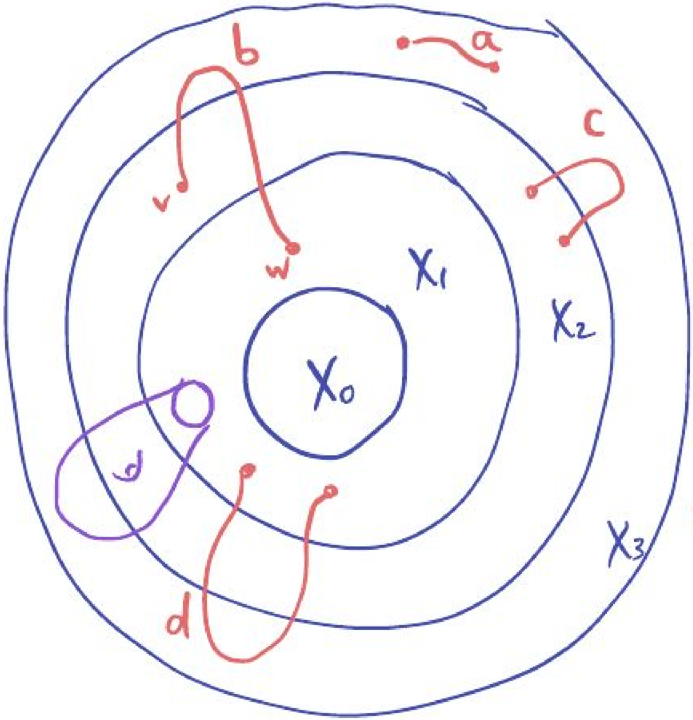
$G_p C_k = F_p C_k / F_{p-1} C_k$

$\partial_p : G_p C_k \longrightarrow G_p C_{k-1}$

In cartoon:

$\partial_3 a \neq 0$ $\partial_3 c = 0$

$\partial_3 b = 0$ $\partial_3 d = 0$.



$\langle v,w \rangle / v \cong \langle v,w \rangle / \langle v-w \rangle$

## More defns

$F_pC_k$ k-chains in $X_p$

$G_pC_k = F_pC_k / F_{p-1}C_k$

$\partial_p : G_pC_k \longrightarrow G_pC_{k-1}$

$E^0_{p,q} = G_pC_{p+q}$

down arrows

$\partial_0 : \underset{p+q \text{ chains}}{E^0_{p,q}} \longrightarrow \underset{p+q-1 \text{ chains}}{E^0_{p,q-1}}$ (usual $\partial$)

$E^1_{p+q} = H_{p+q}(G_pC_*)$

and $\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ ← left arrows.

defined as follows:

given $\alpha \in E^1_{p,q}$, represent it by a chain $x \in F_pC_{p+q} \rightsquigarrow \partial x \in F_pC_{p+q-1}$

$\rightsquigarrow \partial_1(\alpha) = [\partial x]$ ← mod out by $F_{p-1}$

Exercise: $\partial_1$ well def & $\partial_1^2 = 0$

Again: $E^2_{p,q}$ obtained by taking homol.

$$E^2_{p,q} = \frac{\ker(\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q})}{\operatorname{im}(\partial_1 : E^1_{p+1,q} \longrightarrow E^1_{p,q})}$$

We get $E^r_{p,q}$ by repeating the process.

Or a closed formula

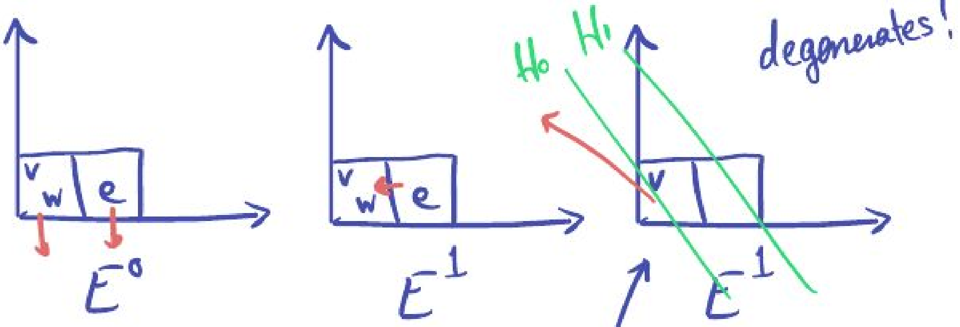
$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} : \partial x \in F_{p-r} C_{p+q-1}\}}{\left(F_{p-1} C_{p+q} + \partial\left(F_{p+r-1} C_{p+q+1}\right)\right) \cap \text{numerator}}$$

# Baby Example 1

$X =$ 

$X_0 = X^{(0)}$

$X_1 = X^{(1)} = X$

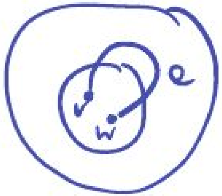


$\Longrightarrow H_0 = \mathbb{Z}$

$H_k = 0 \quad k > 0.$

really

$\langle v, w\rangle / \langle v - w\rangle \cong \langle v\rangle$



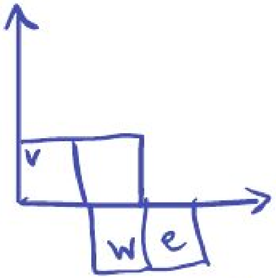
# Baby Example 2

$X =$

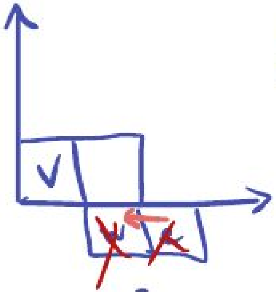
$X_0 = v$

$X_1 = \{v, w\} = X^{(0)}$

$X_2 = X^{(1)} = X$



$E^0 = E^1$



$E^2$

degenerates!

$\Longrightarrow H_0 = \mathbb{Z}$

$H_k = 0 \quad k > 0.$

## Spectral Seq's

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Goal: $H_*(SU(n))$

$X_0 \subseteq X_1 \subseteq \cdots$

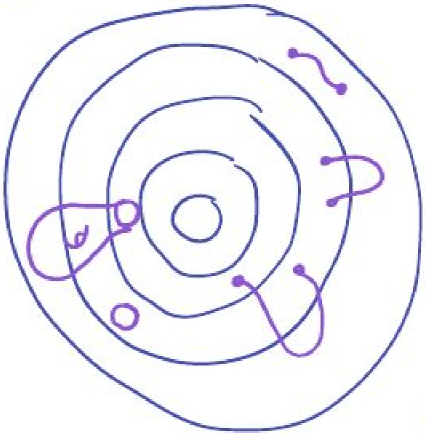
filtration of $X$.

$F_p C_k$ : free abel gp on singular k-chains in $X_p$

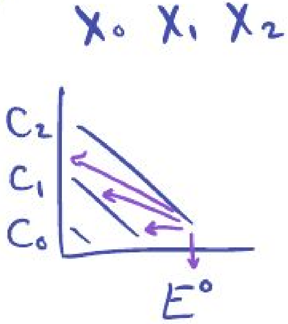
$\rightsquigarrow G_p C_k = F_p C_k / F_{p-1} C_k$

& $\partial_r : G_p C_k \longrightarrow G_{p-r} C_{k-1}$

Cartoon:



$E^0_{p,q} = G_p C_{p+q}$



# TODDLER EXAMPLE

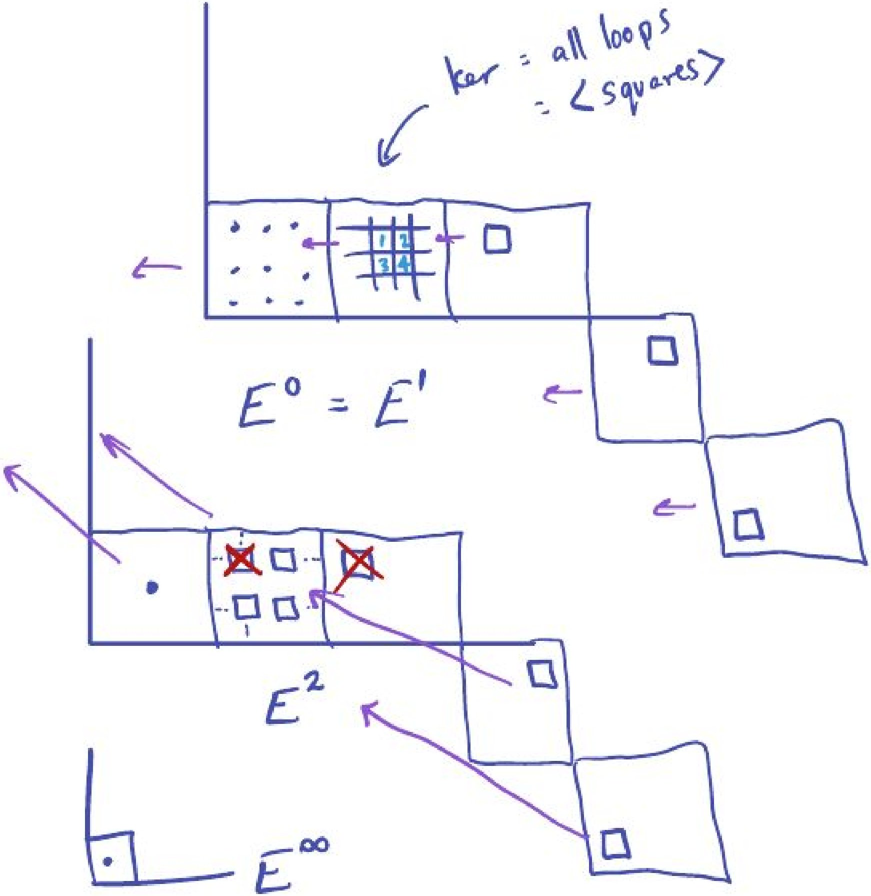
$X = \mathbb{R}^2$ with usual decomp into squares

$X_0 = X^{(0)}$

$X_1 = X^{(1)}$

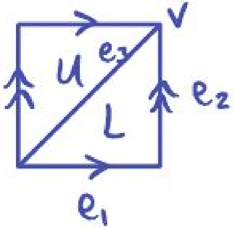
$X_i = X_{i-1} \cup$ one square

$i \geq 2$.

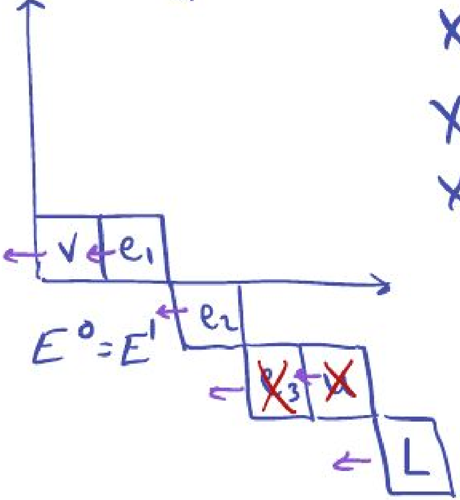


# THE ONE-AT-A-TIME SPECTRAL SEQ

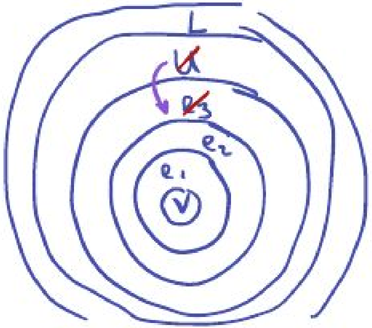
$X =$

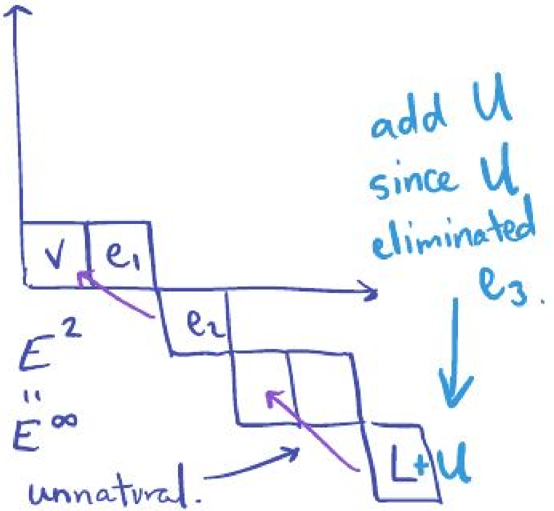


$$X_0 = X^{(0)} = \{v\}$$
$$X_1 = X_0 \cup e_1$$
$$X_2 = X_1 \cup e_2$$
$$X_3 = X_2 \cup e_3$$
$$X_4 = X_3 \cup U$$
$$X_5 = X_4 \cup L$$



$E^0 = E^1$





$E^2 = E^\infty$

add U since U eliminated $e_3$.

$$\Longrightarrow \quad H_0(T) = \mathbb{Z} = \langle v \rangle$$
$$H_1(T) = \mathbb{Z}^2 = \langle e_1, e_2 \rangle$$
$$H_2(T) = \mathbb{Z} = \langle L+U \rangle$$

# APPLICATION: CELLULAR = SINGULAR

**Prop.** For $X$ = cell complex, $H_*(X) = H_*^{cell}(X)$

**Pf.** Use: Spec seq. correctly computes sing. hom $H_*$

Let $X_i = X^{(i)}$

$$\rightsquigarrow E^0_{pq} = C^{sing}_{p+q}(X^{(p)}) / C^{sing}_{p+q}(X^{(p-1)})$$

$$\rightsquigarrow E^1_{pq} = H^{sing}_{p+q}(X^{(p)}, X^{(p-1)})$$

$$= \begin{cases} C^{cell}_p(X) & q=0 \\ 0 & q\neq 0 \end{cases}$$

bottom row of $E^1$

free abel gp on p-cells

Now $(q=0)$

$$\partial_1 : H_p(X^{(p)}, X^{(p-1)}) \to H_{p-1}(X^{(p-1)}, X^{(p-2)})$$

gluing map.

$\Rightarrow E^2$ page is $H_*^{cell}(X)$ in bottom row

$\Rightarrow E^2 = E^\infty$

The prop. follows. □

$$E^0_{pq} = C_{p+q}(X^{(p)})/C_{p+q}(X^{(p-1)})$$

$$\cdot \quad 0$$

$$\begin{tikzcd}
C_0(X^{(0)})/C_0(X^{(-1)}) & C_1(X^{(1)})/C_1(X^{(0)}) \arrow[l] & C_2(X^{(2)})/C_2(X^{(1)}) \arrow[l]
\end{tikzcd}$$

$E^0$

$$\begin{tikzcd}
C_0(X^{(0)})/C_0(X^{(-1)}) & C_1(X^{(2)})/C_1(X^{(1)}) \arrow[l]
\end{tikzcd}$$

$$\leftarrow C_0(X^{(2)})/C_0(X^{(1)})$$

$$\begin{tikzcd} H_0(X^{(0)}, X^{(-1)}) & H_1(X^{(1)}, X^{(0)}) \arrow[l] & \text{etc}\ldots \arrow[l] \end{tikzcd}$$

$E^1$ $\Rightarrow E^2 = E^\infty$ = cell. hom.

# SPECTRAL SEQUENCES

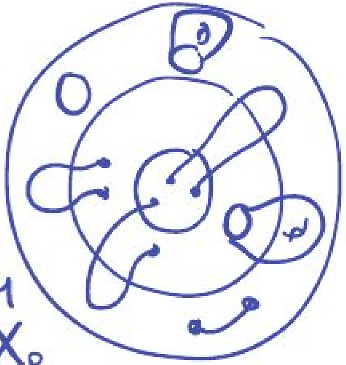
$X_0 \subseteq X_1 \subseteq X_2 \cdots\cdots$

$F_p C_{p+q}$ = abel gp. gen by $p+q$ chains in $X_p$

$G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} : \partial x \in F_{p-r} C_{p+q-1}\}}{(F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1})) \cap \text{numer.}} =$$

$=$ $r^{th}$ approx of cycles. / $r^{th}$ approx of boundary.



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differential $\partial$ is: choose a rep, take boundary. intersect w/ $F_{p-r}C_*$

Thm. $(F_pC_*)$ = filtered complex

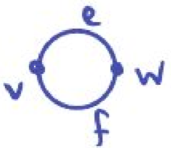
$E^r_{pq}$, $\partial$ as above.

- $\partial_r : E^r_{pq} \longrightarrow E^r_{p-1,q+r-1}$ well def & $\partial_r^2 = 0$.
- $E^{r+1}$ is homology of $(E^r; d_r)$ i.e. $E^{r+1}_{p,q} = \ker d_r / \operatorname{im} d_r$.
- If $C_i$ bdd wrt filtration then $\forall\, p,q\ \exists$ large $r$ s.t. $E^r_{p,q} = G_p H_{p+q} C_*$

Can do w/sing, cell, simpl. homology.

# ONE AT A TIME SPECTRAL SEQ

example

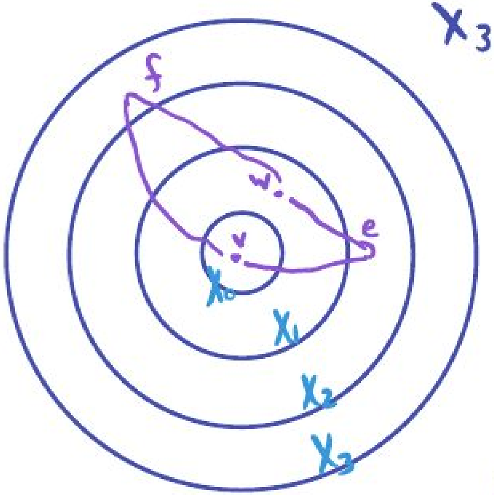


$X_0 = \{v\}$

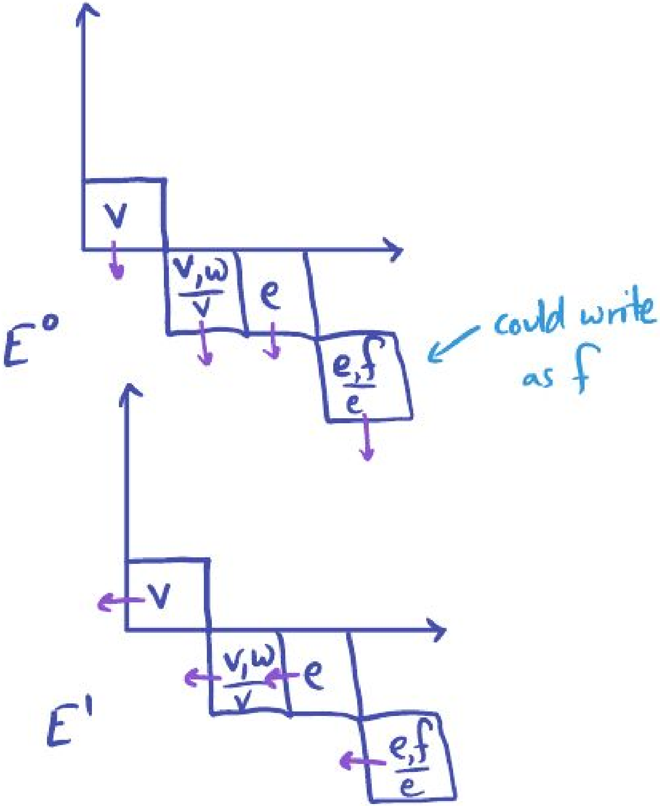
$X_1 = \{v, w\} = X^{(0)}$

$X_2 = \{v, w, e\}$

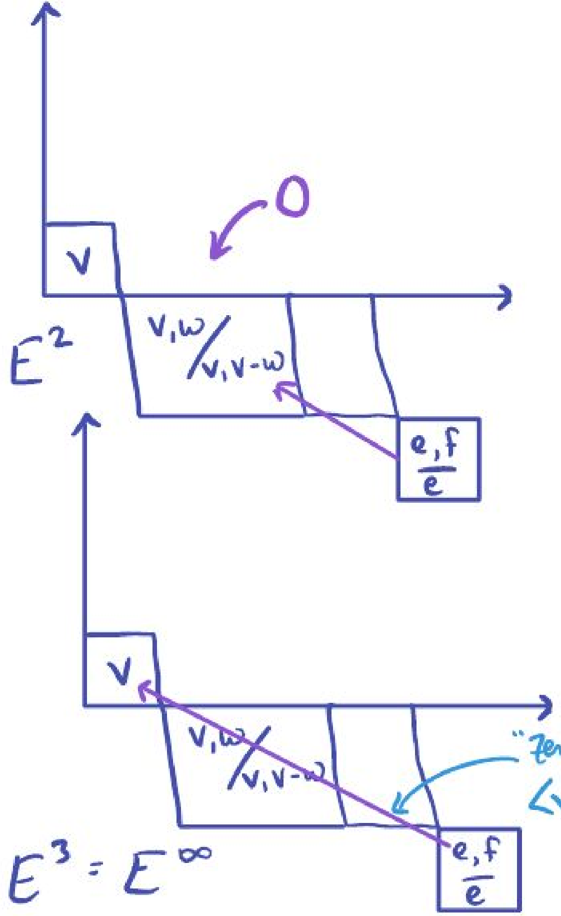
$X_3 = \{v, w, e, f\} = X$



Like the torus example from last time: $\partial e$ is in two different gradings ($X_0$ & $X_1$).

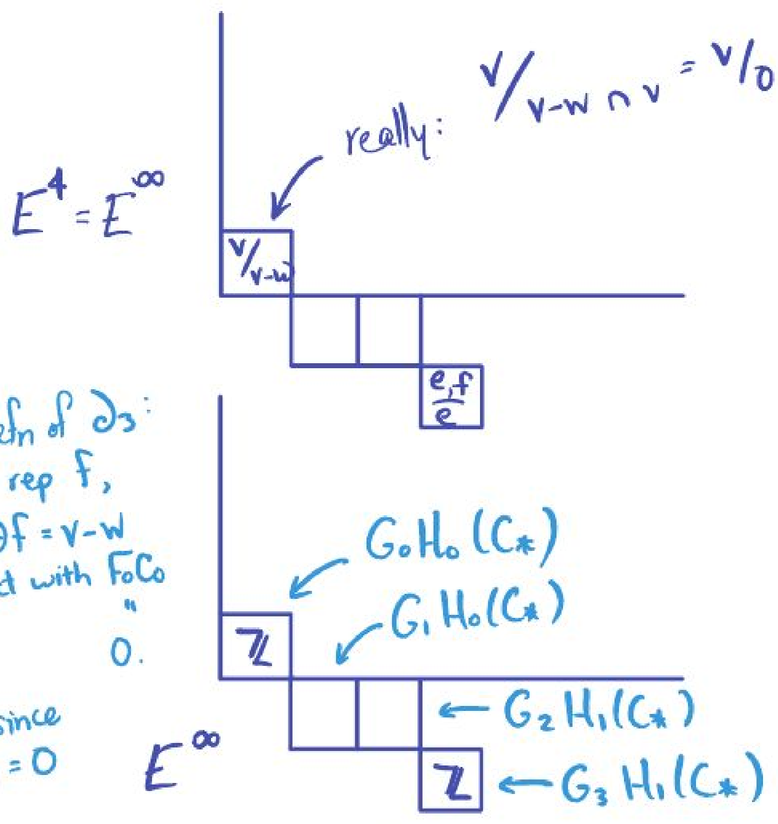


could write as f



by our defn of $\partial_3$: choose rep f, take $\partial f = v-w$ intersect with $F_0C_0$ 〃 0.

"zero map" since $\langle v-w \rangle \cap \langle v \rangle = 0$



really: $V/_{v-w \cap v} = V/_0$

$$\Longrightarrow H_0(X) \cong H_1(X) \cong \mathbb{Z}, \; H_i(X) = 0 \; i > 1$$

## Optional HW

Redo torus
example
from last
time

# SPECTRAL SEQUENCES FOR BEGINNERS

(mostly following Hutchings)

The long exact sequence of a pair allows us to compute $H_*(X)$ in terms of $H_*(A)$ and $H_*(X,A)$.

There is a similar LES for a triple. But what about quadruples, etc.? LES's don't work anymore. The answer is spectral sequences.

## FILTRATIONS

$X$ = CW-complex.
We filter $X$ by subcomplexes: $X_0 \subseteq X_1 \subseteq \cdots$

$\rightsquigarrow$ filtration of $C_*(X)$: $F_p C_k$

$\rightsquigarrow$ associated graded modules:

$$G_p C_k = F_p C_k / F_{p-1} C_k$$

examples ① $X_i$ = $i$-skeleton.
② For a fiber bundle, $X_i$ = pre-image of $i$-skeleton of the base.

# FILTERED CHAIN COMPLEXES

We have $\partial F_p C_k \subseteq F_p C_{k-1}$

$\rightsquigarrow$ induced $\partial : G_p C_k \longrightarrow G_p C_{k-1}$

$\rightsquigarrow$ associated graded chain complex $(G_p C_*, \partial)$

and: induced filtration on $H_*(X)$:

$$F_p H_k(X) = \{ \alpha \in H_k(X) : \exists \; x \in F_p C_k \text{ s.t. } \alpha = [x] \}$$

$\rightsquigarrow$ associated graded pieces $G_p H_k(X)$.

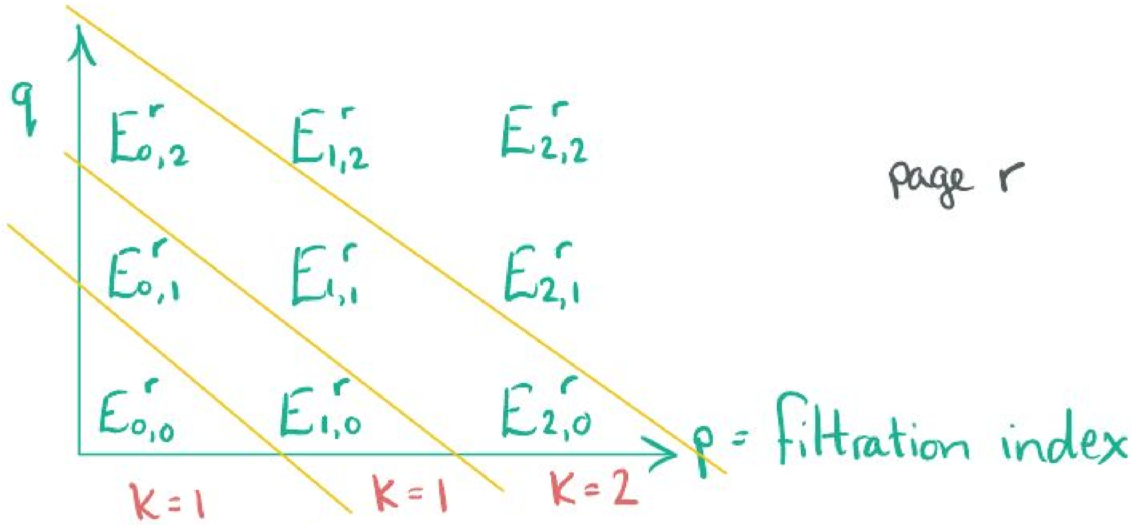
Hope. $H_*(G_p C_*)$ is easy to compute and it determines $G_p H_*(C_*)$, hence $H_*(X)$. We know it works for $\emptyset \subseteq A \subseteq X$.

Will compute $H_*(X)$ by "successive approximations"

# OVERVIEW

A spectral sequence has pages. Each page is a 2D grid of vector spaces (let's work over a field). There are also differentials, and we get from one page to the next by taking homology.

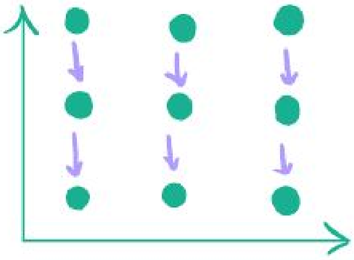
Each page looks like:



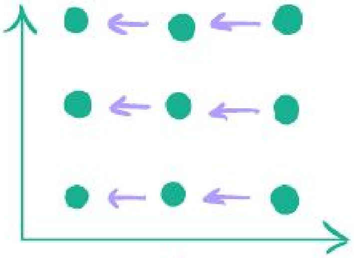
The $E^r_{p,q}$ with $p+q=k$ correspond to k-chains at the various levels of the filtration.

e.g. $$E^0_{p,q} = G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$$

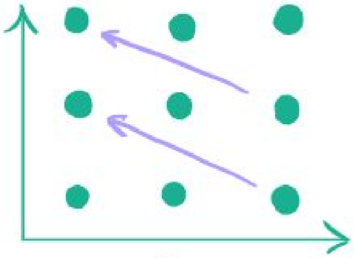
The differentials always reduce dimension by 1, but as r increases they go further down the filtration. Specifically, on page r, differentials go r units left and r-1 units up.



$E^0$



$E^1$



$E^2$

In favorable cases, each term $E^r_{p,q}$ stabilizes with r. For instance if the $E^0_{p,q}$ are 0 outside the first quadrant (all the differentials are eventually 0). We define $E^\infty_{p,q}$ to be this term. The $\infty$ page is made of these terms.

Think about paintball. Each generator for $E^0_{p,q}$ gets a paintball. When someone shoots a paintball, both the target and the shooter get eliminated.

We will see: $E^\infty_{p,q} = G_p H_{p+q}(C_*)$

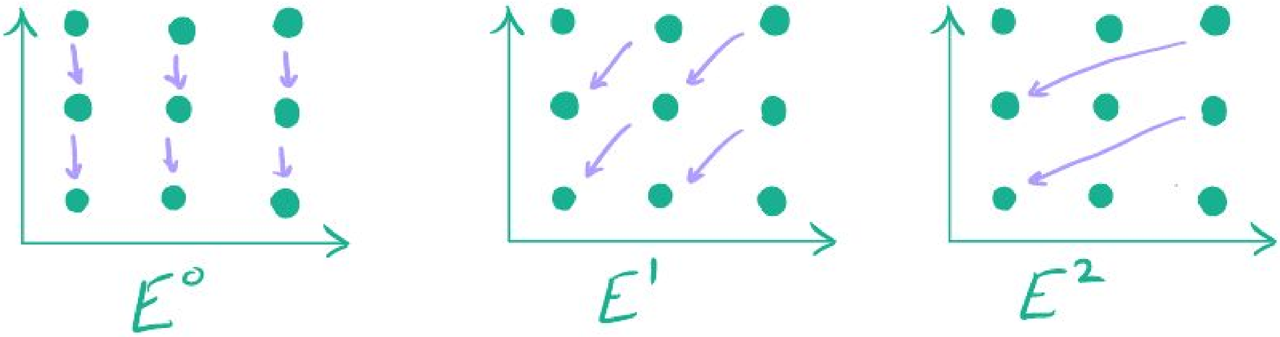
Sometimes a spectral sequence degenerates, which means all terms stabilize at the same time.

# INDEXING (AN ASIDE)

The indexing probably seems weird. Also, the way the arrows turn might seem mysterious. If we instead choose the obvious indexing:

$$E^0_{p,q} = G_p C_q$$

then the arrows are more natural:



A downside is that for most natural filtrations, the bottom right of the 1st quadrant would be 0's.

Also, Serre invented spectral sequences for fibrations. There, $E^2_{p,q} = H_p(B; H_q(F))$, which is nice!

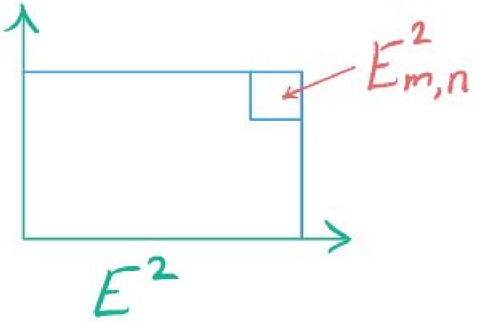
By the way, Serre's result illustrates the general pattern. If a theorem starts with "There is a spectral sequence..." then often what the theorem does is describe the $E^2$ page.

# USING SPECTRAL SEQUENCES

Let's say a word about using spectral sequences (yes, before we formally say what they are!)

Often, when using a long exact sequence, the hope is that there are lots of zeros. For instance, if every third term is 0, the remaining maps are isomorphisms.

It's the same with spectral sequences. Here's an example. We said that in Serre's spectral sequence we have $E^2_{p,q} = H_p(B, H_q(F))$. So if $B$ is m-dimensional and $F$ is n-dimensional, the $E^2$ page lives in the $m \times n$ rectangle:



All arrows going in & out of $E^r_{m,n}$ are 0 for $r \geq 2$. So: $E^2_{m,n} = E^\infty_{m,n} \cong H_{m+n}(E)$.

# FORMAL DEFINITIONS AND STATEMENTS

Say we have the $X_p$, $F_p C_*$, $G_p C_*$ as above.
We set $E^0_{p,q} = G_p C_{p+q}$
$\partial_0 : E^0_{p,q} \to E_{p,q-1}$ (= usual boundary $\partial$)

Then $E^1_{p,q}$ is obtained by taking homology at $E^0_{p,q}$, so $E^1_{p,q} = H_{p+q}(G_p C_*)$

& $\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ is defined as:
given $\alpha \in E^1_{p,q}$, represent it by a chain
$x \in F_p C_{p+q} \rightsquigarrow \partial x \in F_p C_{p+q-1}$
$\rightsquigarrow \partial_1(\alpha) = [\partial x]$.

In other words $\partial_1$ is the usual $\partial$ in the same sense as $\delta : H_n(X,A) \longrightarrow H_{n-1}(A)$ is the usual $\partial$.

Exercise: $\partial_1$ is well def. & $\partial_1^2 = 0$.

Again, $E^2_{p,q}$ obtained by taking homology:

$$E^2_{p,q} = \frac{\ker(\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q})}{\operatorname{im}(\partial_1 : E^1_{p+1,q} \longrightarrow E^1_{p,q})}$$

In general: $E^r_{p,q} = \dfrac{\{x \in F_p C_{p+q} : \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})}$

where really we quotient by the intersection of the denominator by the numerator.

This is an approximation of cycles/boundaries: if a chain has boundary, but the boundary is far down the filtration, we consider it a cycle (for now). Similarly, if a chain is a boundary of a chain much higher in the filtration, we consider it to not be a boundary (for now).

**Proposition.** Let $(F_p C_*, \partial)$ be a filtered complex, and define the $E^r_{p,q}$ as above. Then:

① $\partial$ induces a well-defined map

$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r, q+r-1} \quad \text{with } \partial_r^2 = 0.$$

② $E^{r+1}$ is the homology of $(E^r, \partial_r)$.

③ $E^1_{p,q} = H_{p+q}(G_p C_*)$

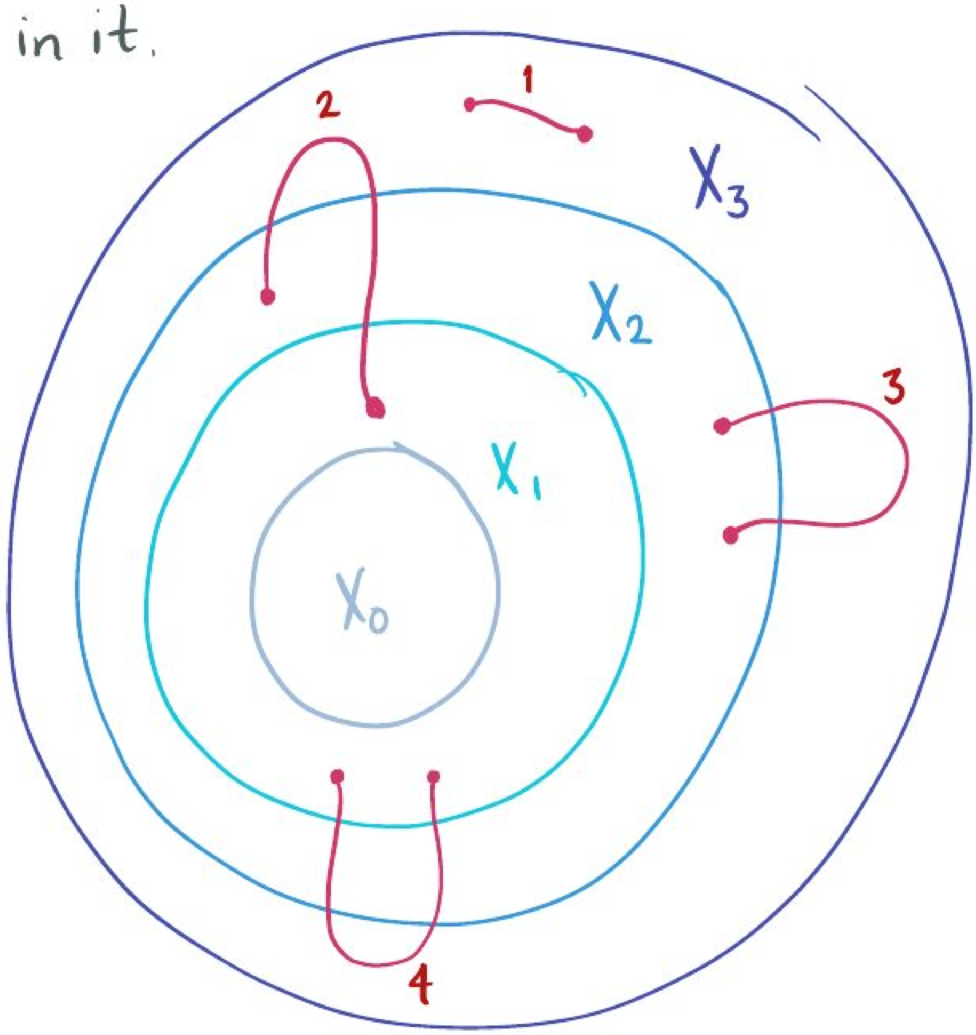
④ If the filtration of $C_i$ is bounded $\forall i$ then $\forall p,q$ if $r$ is sufficiently large then

$$E^r_{p,q} = G_p H_{p+q}(C_*)$$

**Pf.** Exercise

# CARTOON

Here is a schematic of a filtration, and some chains in it.



So the edge 2 lies in $X_3$, but its boundary lies in $X_2$, and one component of the boundary lies in $X_1$.

Zeroth approximation: Take boundaries in $X_p/X_{p-1}$
So a chain in $X_p$ is a cycle if its boundary lies in $X_{p-1}$. In this approximation, the edge labeled 1 is not a cycle but the others are.

First approximation: Of the remaining chains, see if they have boundary in $X_{p-1}/X_{p-2}$, etc.

The edges labeled 2 and 3 have boundary in the 1st approximation.
The edge labeled 4 has boundary in the 2nd approx.

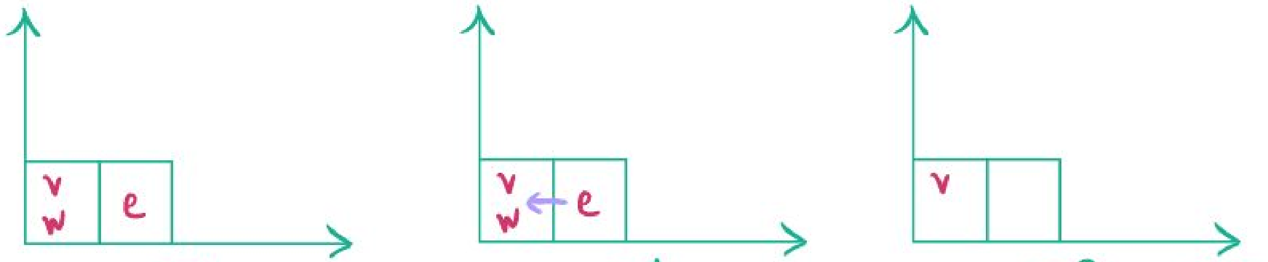
At each stage we take homology, so at the stage when we discover a chain's boundary, the boundary gets killed and the chain with boundary gets forgotten since it is not a cycle.

(Can think of searching for each chain's boundary with a stronger & stronger flashlight.)

These stages are exactly the pages of the spectral sequence.

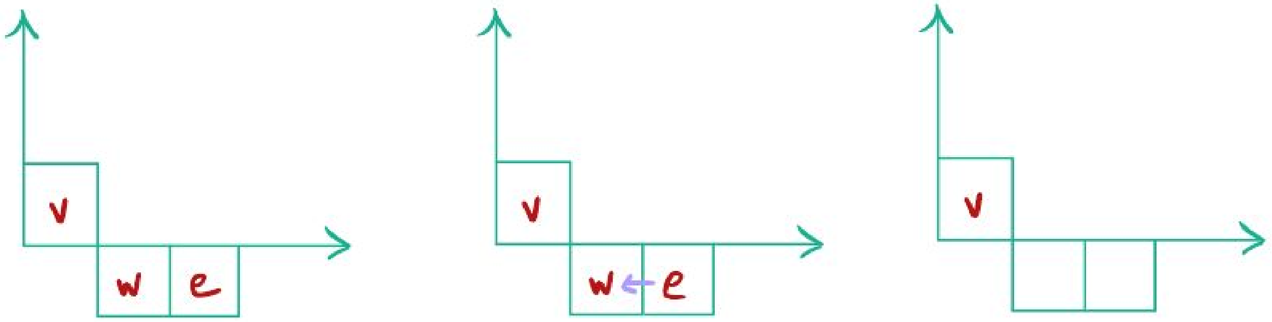
# BABY EXAMPLES

**Example 1.** $X = \overset{v}{\bullet}\overset{e}{\rule{1cm}{0.4pt}}\overset{w}{\bullet}$ , $X_0 = X^{(0)}$ , $X_1 = X^{(1)} = X$ .



$E^0$ $E^1$ $E^2 = E^\infty$

**Example 2.** $X = \overset{v}{\bullet}\overset{e}{\rule{1cm}{0.4pt}}\overset{w}{\bullet}$ , $X_0 = \{v\}$, $X_1 = \{v, w\}$, $X_2 = X$ .



$E^0$ $E^1$ $E^2 = E^\infty$

Of course we get that $H_0(X; F) = F$ both times.
The first spectral sequence gives

$$H_0(X; F) = \langle v, w \rangle / \langle v - w \rangle$$

and the second gives: $H_0(X; F) = \langle v, w \rangle / \langle w \rangle$

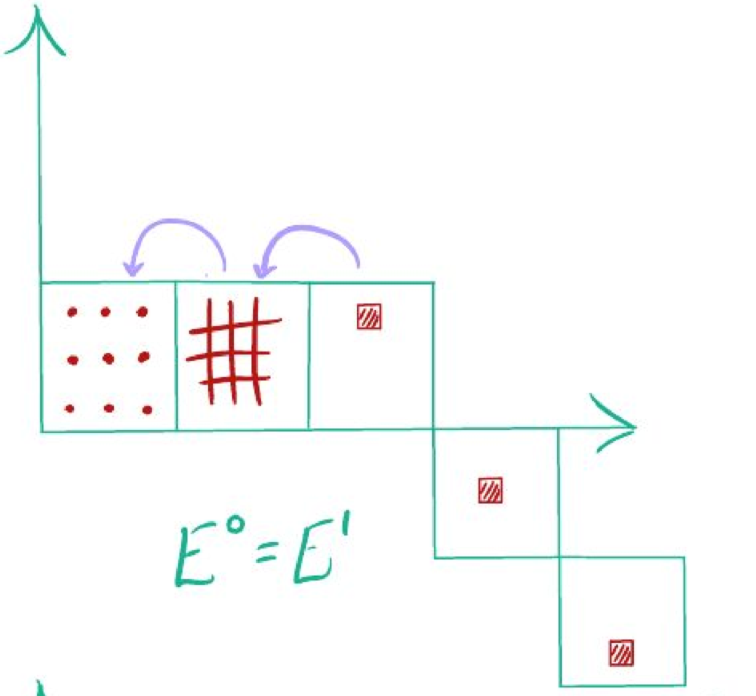
# TODDLER EXAMPLE

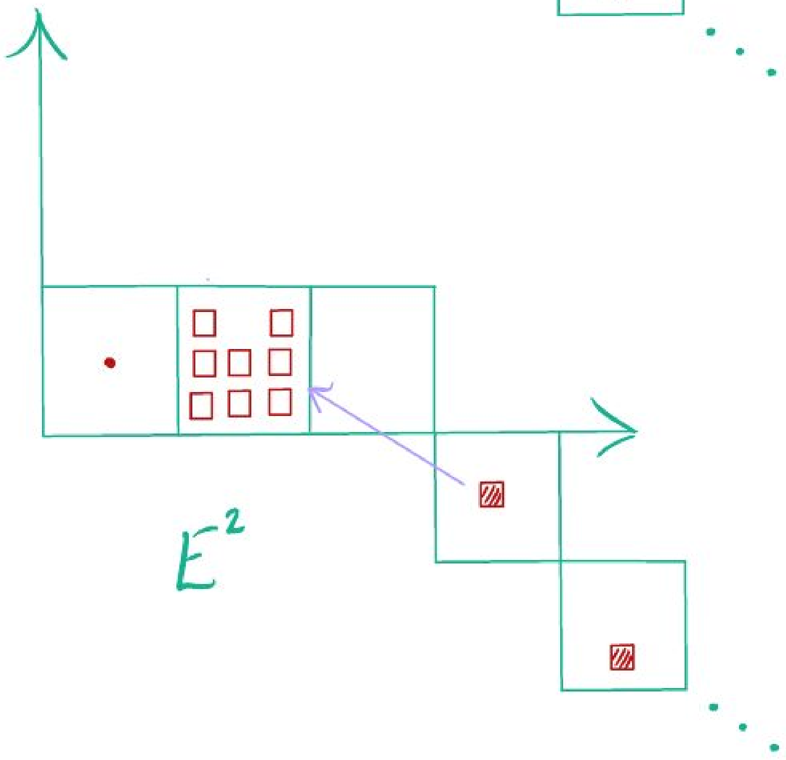
Example 3. $X = \mathbb{R}^2$ with usual cell decomp. into unit squares.

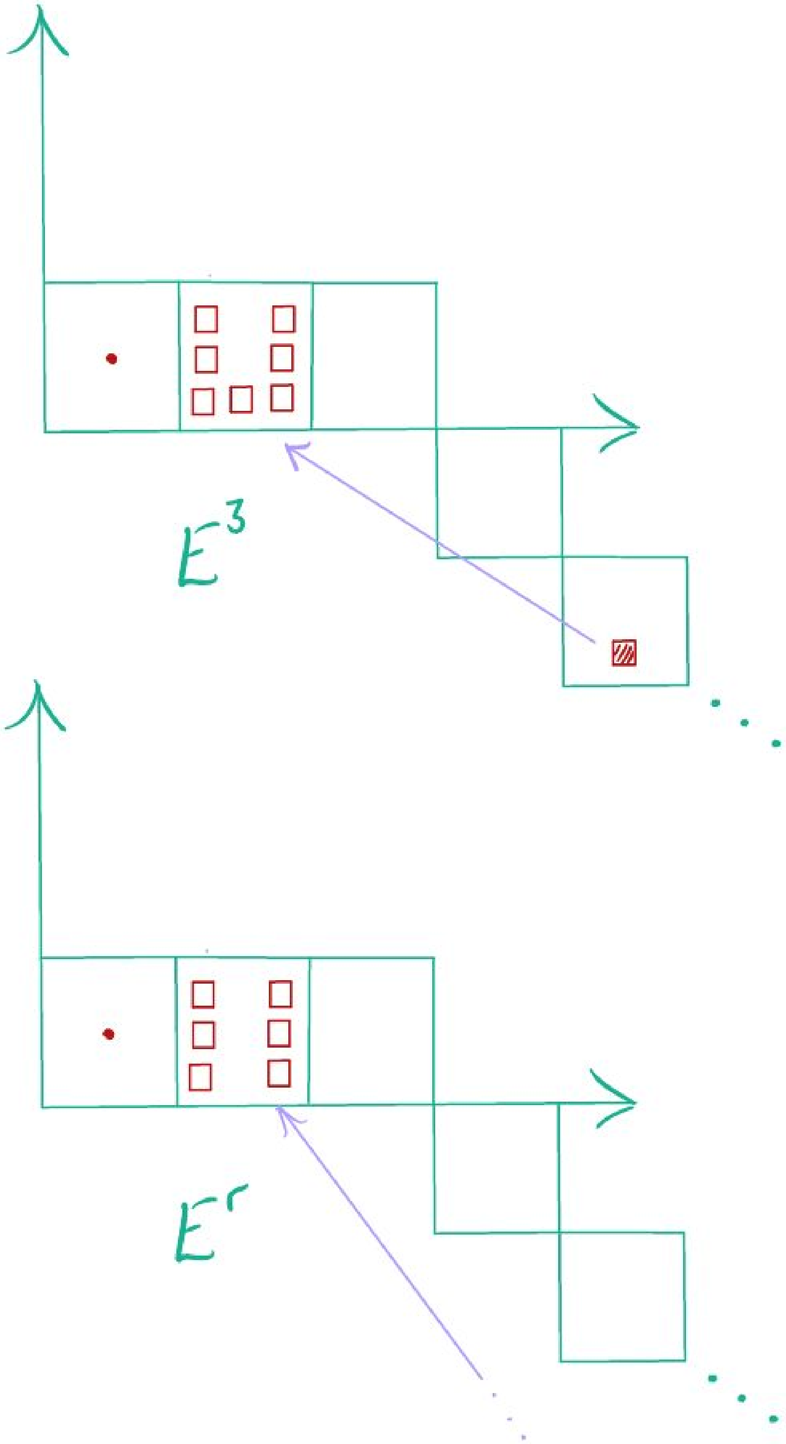
$X_0 = X^{(0)}$

$X_1 = X^{(1)}$

$X_i = X_{i-1} \cup \{\text{one square}\} \quad i \geq 2$





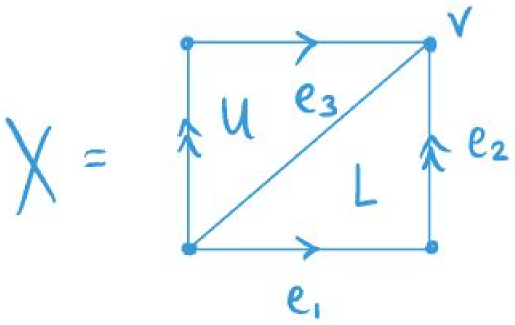


Eventually, all the squares get killed.

This filtration is not bounded, so you'll need to think about direct limits (or do a finite grid instead)

# THE ONE-AT-A-TIME SPECTRAL SEQUENCE

Similar to the last example, lets compute the homology of $T^2$ by adding one cell at a time. Use $\mathbb{Z}/2$



$X_0 = \{v\}$

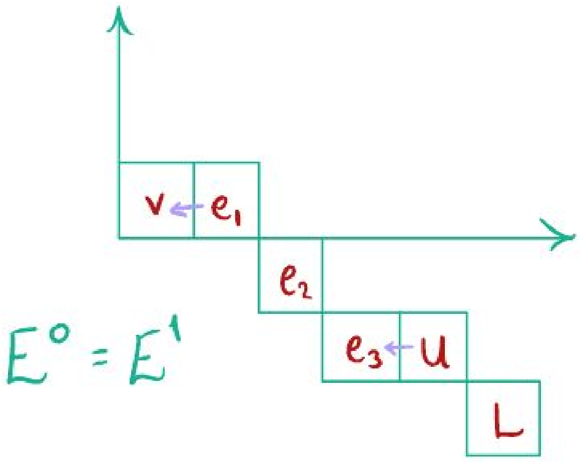
$X_1 = X_0 \cup e_1$

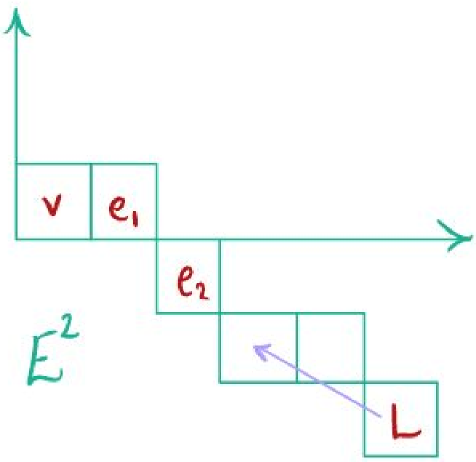
$X_2 = X_1 \cup e_2$

$X_3 = X_2 \cup e_3$

$X_4 = X_3 \cup U$

$X_5 = X_4 \cup L$





Here we have a new phenomenon we didn't see in the last example. The cell L does have boundary in $F_3 C_*$, namely $e_3$. But $e_3$ has already been eliminated. The natural way to handle this is to add U to L, since U is the cell that eliminated $e_3$. This is ok, since the $E^r_{pq}$ are all quotients. If we do this, we get that $E^2 = E^\infty$ and that $H_*(T^2)$ is generated by $v$, $e_1$, $e_2$, & $L+U$, as usual.

# APPLICATION: CELLULAR = SINGULAR

**Prop.** For $X$ a $\Delta$-complex, $H_*(X) \cong H_*^{cell}(X)$

**Pf.** Let $X_i = X^{(i)}$ (filtration by skeleta).

$\rightsquigarrow E^0_{pq} = C_{p+q}(X^{(p)})/C_{p+q}(X^{(p-1)})$

$\rightsquigarrow E^1_{pq} = H_{p+q}(X^{(p)}, X^{(p-1)})$ (by defn of rel. hom.)

Recall: $$H_{p+q}(X^{(p)}, X^{(p-1)}) \cong \begin{cases} C_p^{cell}(X) & q=0 \\ 0 & q \neq 0 \end{cases}$$

where $C_p^{cell}(X)$ is the free $F$-module on the $p$-cells.

Now: $\partial_1 : H_p(X^{(p)}, X^{(p-1)}) \longrightarrow H_{p-1}(X^{(p)}, X^{(p-1)})$ is the usual $\partial$ (cf. LES for triple).

This exactly records the gluing maps of the $p$-cells to the $(p-1)$-skeleton.

$\Rightarrow E^2$ page is $H_*^{cell}(X)$ in bottom row, and $0$ elsewhere

$\Rightarrow E^\infty = E^2$ (the spec. seq. degenerates on page 2).

The proposition follows. ▨

# APPLICATION: KÜNNETH

$(C_*, \partial)$ , $(C'_*, \partial')$ chain complexes over a field

$(C \otimes C')_k = \bigoplus_{i+j=k} C_i \otimes C'_j$

and $\partial(\alpha \otimes \beta) = (\partial\alpha) \otimes \beta + (-1)^i \alpha \otimes (\partial'\beta)$ $\alpha \in C_i, \beta \in C'_j$

**Prop.** The natural map

$$\bigoplus_{i+j=k} H_i(C_*) \otimes H_j(C'_*) \longrightarrow H_{i+j}(C \otimes C')$$

is an isomorphism.

**Pf.** Define $F_p(C \otimes C')_k = \bigoplus_{i \leq p} C_i \otimes C_{k-i}$

$$\rightsquigarrow E^0_{p,q} = G_p(C \otimes C')_{p+q} = C_p \otimes C'_q$$

Have
$$\begin{aligned} \partial(C_p \otimes C'_q) &\subseteq (\partial C_p \otimes C'_q) \oplus (C_p \otimes \partial' C'_q) \\ &\subseteq (C_{p-1} \otimes C'_q) \oplus (C_p \otimes C'_{q-1}) \\ &\subseteq G_{p-1} \oplus G_p \end{aligned}$$

So we already see that the spectral sequence will degenerate on page 2. The differential only reaches down one level of the filtration.

From above: $\partial_0 = (-1)^p \otimes \partial'$

We want $E^1_{pq} = {}^{\ker \partial_0}/_{\mathrm{im}\,\partial_0}$. Note the $(-1)^p$ does not affect the kernel or the image.

$\rightsquigarrow$ $E^1_{pq}$ is the homology of the chain complex

$$\cdots \to C_p \otimes C'_{q+1} \xrightarrow{\partial'} C_p \otimes C'_q \to C_p \otimes C'_{q-1} \to \cdots$$

which is, by definition: $H_*(C'_*; C_p)$.

The universal coefficient theorem for homology:

$$0 \to H_n(C'_*) \otimes C_p \to H_n(C'_*; C_p) \to \mathrm{Tor}(H_{n-1}(C'_*), C_p) \to 0$$

But $\mathrm{Tor}(A,B) = 0$ if $A$ or $B$ is torsion free

$$\Longrightarrow H_*(C'_*; C_p) \cong H_*(C'_*) \otimes C_p$$

So $E^1_{pq} \cong C_p \otimes H_q(C'_*)$

Next $\partial_1 = \partial \otimes 1$. Similar as above, $E^2_{pq}$ is the homology of

$$\cdots \to C_{p+1} \otimes H_q(C'_*) \to C_p \otimes H_q(C'_*) \to C_{p-1} \otimes H_q(C'_*) \to \cdots$$

We are working over a field. So the $H_q(C'_*)$ are torsion free

$\rightsquigarrow$ can apply UCT as above

$\rightsquigarrow$ $E^2_{pq} = H_p(C_* \otimes H_q(C'_*)) = H_p(C_*) \otimes H_q(C'_*)$

Each elt of $E^2_{pq}$ is represented by $\alpha \otimes \beta$ where $\alpha$ is a cycle in $C_p$ & $\beta$ is a cycle in $C'_q$.

$\Rightarrow$ $\alpha \otimes \beta$ is a cycle in $C_* \otimes C'_*$.

$\Rightarrow$ all higher differentials vanish, ie. $E^2 = E^\infty$.

The proposition follows ▨

For the Künneth formula, you also want to know that $H_*(X \times Y) \cong H_*(C_*(X) \otimes C_*(Y))$, but this is straightforward with simplicial homology.

# FIBER BUNDLES

Next goal: Leray-Serre spectral sequence for fiber bundles.

A fiber bundle is a space that locally looks like a product (perhaps not globally).

First examples: cylinder, Möbius band are $[0,1]$-bundles over $S^1$.

Definition. $B$ = connected space, $b_0 \in B$ base point
A continuous map $\pi: E \to B$ is a fiber bundle with fiber $F$ if $\forall x \in B$ $\exists$ open nbd $U$ & $\psi_U$ as follows:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & U \times F \arrow[dl] \\
U &
\end{tikzcd}$$

fiber  total space

Write: $$\begin{tikzcd}
F \arrow[r] & E \arrow[d] \\
 & B
\end{tikzcd}$$

base

# EXAMPLES

0. Trivial bundle $E = F \times B$.
1. Covering spaces. $F =$ discrete set.
2. Cylinder & Möbius band. $F = I$, $B = S^1$
3. Torus & Klein bottle $F = S^1$, $B = S^1$
4. Vector bundles, e.g. tangent bundle
5. Sphere bundles, e.g. unit tangent bundle.
   Hopf fibration $\rightsquigarrow \pi_3(S^2) \neq 0$.
6. Mapping torus $B = S^1$.
7. Lie groups. $G =$ Lie group, $H =$ compact subgroup

$$\begin{tikzcd} H \arrow[r] & G \arrow[d] \\ & G/H \end{tikzcd}$$

   In fact this is a principal $H$-bundle : $H$ acts in a fiberwise way on $E = G$.

8. More Lie groups. $E =$ smooth manifold.
   $G =$ compact Lie gp
   $G \circlearrowright E$ freely, smoothly
   $\rightsquigarrow E \to E/G$

Basic problems: classify bundles, understand sections
(Hairy ball theorem is a section problem.)

# UNITARY GROUPS

Inner product on $\mathbb{C}^n$ : $\langle u,v\rangle = \sum u_i \bar{v}_i$

$U(n) = \{M \in GL_n\mathbb{C} : M \text{ preserves } \langle\, ,\rangle\}$

$SU(n) = \{M \in U(n) : \det(M) = 1\}$

**Prop.** We have a fiber bundle

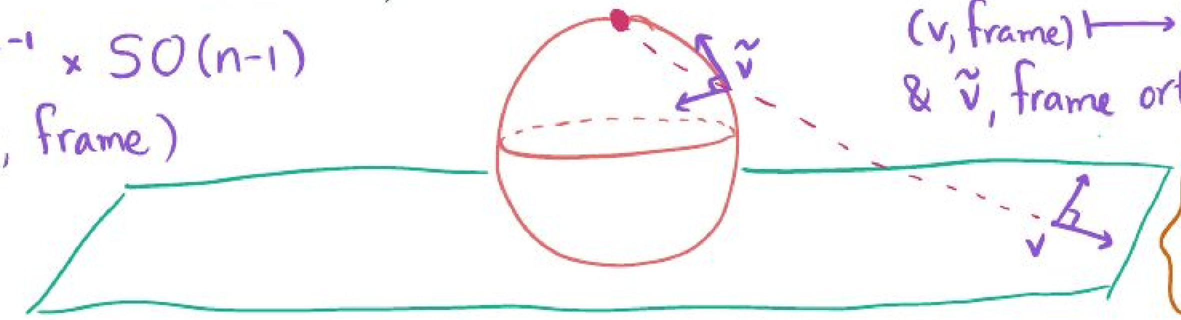
$$\begin{tikzcd} SU(n-1) \arrow[r] & SU(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

**Proof #1.** $SU(n-1)$ compact subgp of Lie gp $SU(n)$. So suffices to show $SU(n)/SU(n-1) \cong S^{2n-1}$. $SU(n)$ acts transitively on unit sphere in $\mathbb{C}^n$, namely, $S^{2n-1}$. Stabilizer of a point is $U(n-1)$, e.g. stabilizer of $e_n$ is

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \qquad A \in SU(n-1)$$

**Proof #2.** (O(n) version) Stereographic projection is conformal. So the inverse maps the trivial $SO(n-1)$-bundle over $\mathbb{R}^{n-1}$ to the trivial $SO(n-1)$-bundle over $S^{n-1} \setminus$ north pole.

$\mathbb{R}^{n-1} \times SO(n-1)$
(pt, frame)



$(v, \text{frame}) \longmapsto (\tilde{v}, \text{frame})$
& $\tilde{v}$, frame orthonormal.

Same for $SU(n)$

For $n=3$:

$$\begin{tikzcd}
SU(1) \arrow[r] \arrow[d, equal] & SU(2) \arrow[d] \\
\{1\} & S^3
\end{tikzcd}$$

$$\Longrightarrow SU(2) \cong S^3$$

Another way to see this:

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

The equation $|\alpha|^2+|\beta|^2=1$ gives unit sphere in $\mathbb{C}^2$.

Also, $SU(2) = \{\text{unit quaternions}\}$

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

We will use the Serre spectral sequence to compute $H_*(SU(n))$ for $n=3,4$. (Note $H_*(SO(n))$ is already computed in Sec. 3D of Hatcher, using an explicit cell decomposition.)

Part of the point is to show off spectral sequences as a microwave oven — often you can get something useful out with little effort or deep knowledge of the inner workings.

# SERRE SPECTRAL SEQUENCE

Thm. Let $E \to B$ be a fiber bundle with fiber F. Then there is a spectral sequence $E^r_{pq}$ with

$$E^2_{pq} = H_p(B; \{H_q(E_x)\})$$

and converging to:

$$E^\infty_{pq}\ G_p H_{p+q}(E)$$

for some filtration on $H_*(E)$.

Note: The coefficients here are <u>local</u>. Local coefficients are the same as constant coefficients when $\pi_1(B) = 1$.

Local Coefficients. $\pi = \pi_1(X)$, $M = \mathbb{Z}[\pi]$-module
$\tilde{X}$ = universal cover.

Then $H_*(X; \{M\})$ is the homology of

$$C_n(\tilde{X}) \otimes_\pi M$$

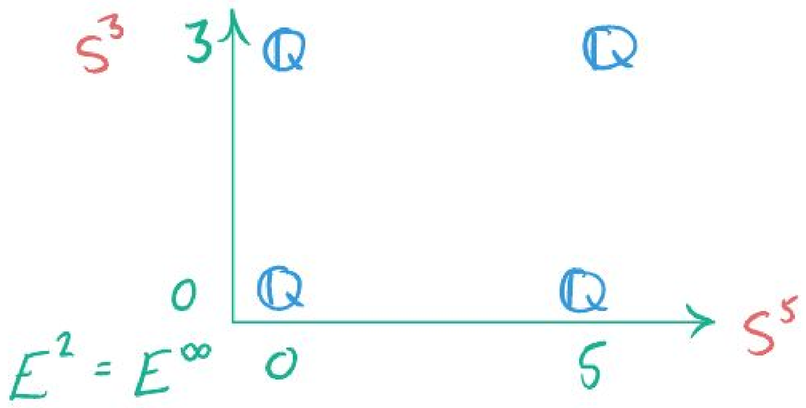
really this's $\mathbb{Z}[\pi]$ but we emphasize the $\pi$

For two <u>left</u> modules $A, B$ over a ring $R$, $A \otimes_R B$ is the abelian group gen by $\{a \otimes b\}$ subject to distributivity and: $a \otimes b = ra \otimes rb$ (ie factor out by $R$-action).

# APPLICATION TO SU(n)

Let's compute $H_*(SU(3))$.

$$\begin{tikzcd} SU(2) \arrow[r] \arrow[d, equal] & SU(3) \arrow[d] \\ S^3 & S^5 \end{tikzcd}$$

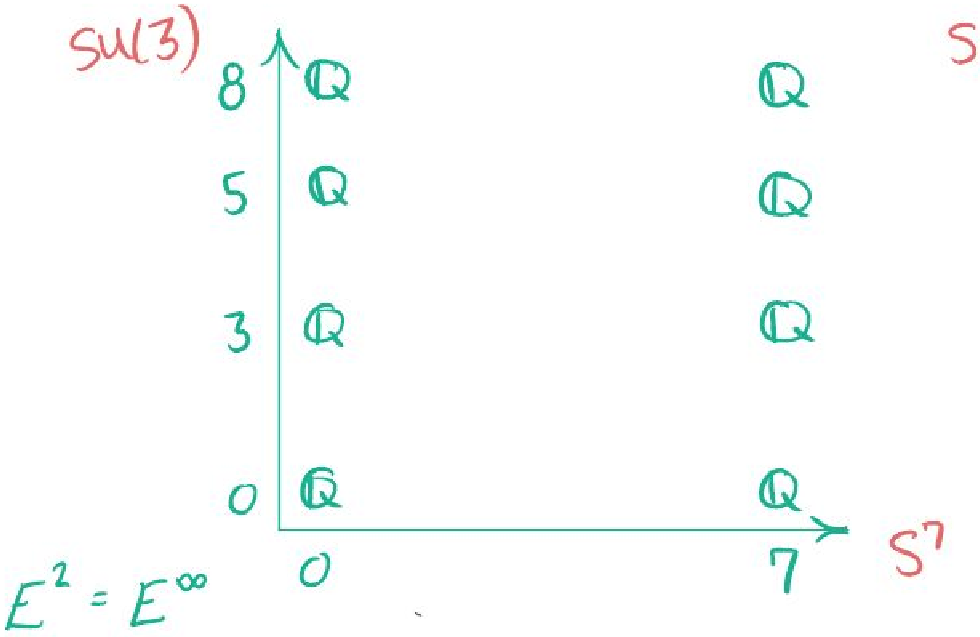


$$\Rightarrow H_k(SU(3)) = \begin{cases} \mathbb{Q} & k = 0, 3, 5, 8 \\ 0 & \text{otherwise.} \end{cases}$$

$$= H_k(S^3 \times S^5)$$

...And $H_*(SU(4))$

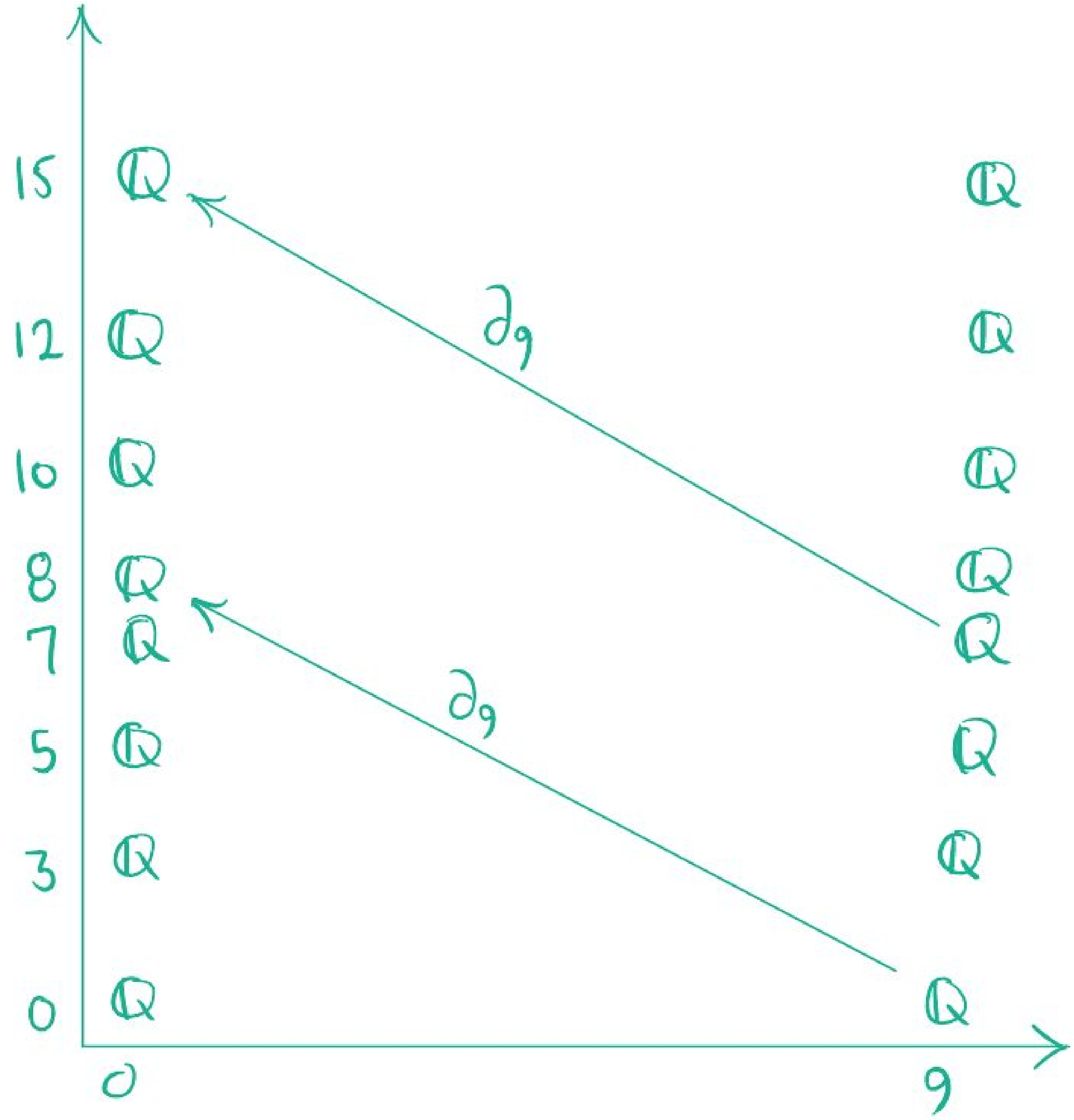
$$\begin{tikzcd} SU(3) \arrow[r] & SU(4) \arrow[d] \\ & S^7 \end{tikzcd}$$



$$\Rightarrow H_k(SU(4)) = \begin{cases} \mathbb{Q} & k = 0, 3, 5, 8, 10, 12, 15 \\ 0 & \text{otherwise.} \end{cases}$$

$$= H_k(S^3 \times S^5 \times S^7)$$

Unfortunately for $SU(5)$ there are differentials to consider.



But they turn out to be zero!

<u>Thm.</u> $H_*(SU(n)) = H_*(S^3 \times S^5 \times \cdots \times S^{2n-1})$

These spaces are not homotopy equivalent!

# AN EXAMPLE WITH NONTRIVIAL COEFFICIENTS

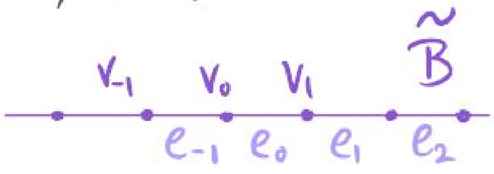
Lets compute $H_*$ of $X$ = Klein bottle with Serre:

$B = S^1$ $F = S^1$, coefficients $M = \mathbb{Z}$ or $\mathbb{Z}/2$

$$\begin{array}{c|ll} F & & \\ & H_0(B;\, H_1(F;M)) & H_1(B;\, H_1(F;M)) \\ & H_0(B;\, H_0(F;M)) & H_1(B;\, H_0(F;M)) \\ \hline E^2 & & \rightarrow B \end{array}$$

The spectral seq. is degenerate, so it remains to compute the homology gps (and solve the extension problem).

Denote generators for $\pi_1(B)$ & $H_1(F;M)$ by $b, f$.
The action $\pi_1(B) \circlearrowright H_k(F)$ is trivial for $k=0$
and given by $b \cdot f = -f$.
So bottom row has trivial (not local) coefficients.



Let's compute $H_*(B;\, H_1(F;M))$.
First, $C_0(\widetilde{B}) \otimes H_1(F;M)$ is gen. by $v_i \otimes f$
subject to $v_i \otimes f = bv_i \otimes b \cdot f = v_{i+1} \otimes -f = -v_{i+1} \otimes f$
$\rightsquigarrow$ it is gen. by $v_0 \otimes f$

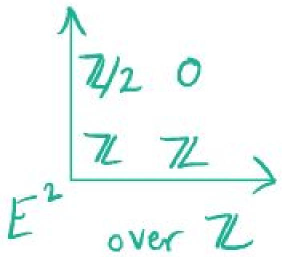
Similarly, $C_1(\widetilde{B}) \otimes H_1(F;M)$ is gen by $e_0 \otimes f$

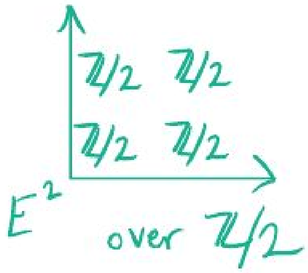
$\rightsquigarrow$ chain complex

$$0 \longrightarrow C_1(\tilde{B}) \otimes H_1(F;M) \xrightarrow{\partial \otimes 1} C_0(\tilde{B}) \otimes H_1(F;M) \longrightarrow 0$$

$$\begin{aligned} e_1 \otimes f \longmapsto (v_1 - v_0) \otimes f \\ = v_1 \otimes f - v_0 \otimes f \\ = -2v_0 \otimes f \end{aligned}$$

$$\Longrightarrow \quad H_1(B; H_1(F;\mathbb{Z})) = 0 \qquad H_0(B; H_1(F;\mathbb{Z})) = \mathbb{Z}/2$$
$$H_1(B; H_1(F;\mathbb{Z}/2)) = \mathbb{Z}/2 \qquad H_0(B; H_1(F;\mathbb{Z}/2)) = \mathbb{Z}/2.$$





This agrees with what we know:

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & k=1 \\ 0 & k>1 \end{cases} \qquad H_k(X;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & k=0,2 \\ (\mathbb{Z}/2)^2 & k=1 \\ 0 & k>2 \end{cases}$$

For $H_1(X;\mathbb{Z})$ have: $0 \longrightarrow \mathbb{Z} \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$.
Need to verify this is the trivial extension.

# INSIDE THE SERRE S.S.

Let $B^p$ = p-skeleton of $B$.

$F_p C_*(E)$ = singular chains supported in $\pi^{-1}(B^p)$.

$\rightsquigarrow G_p C_*(E) = C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$

$\longrightarrow E^1_{pq} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$

Can calculate as a direct sum over p-cells $\sigma : D^p \longrightarrow B$ of $H_{p+q}$ of pullback bundle:

$$E^1_{pq} = \bigoplus_{\sigma} H_{p+q}(\sigma^* E, (\sigma^* |_{S^{p-1}})^* E)$$

↖ pullback bundles

$$= \bigoplus_{\sigma} H_{p+q}(D^p \times F, S^{p-1} \times F)$$

← bundles over simply connected spaces are trivial.

$$= \bigoplus_{\sigma} H_q(F)$$

See Fomenko p. 140.

**Claim.** The latter is $C_p^{cell}(B; H_q(F))$

**Pf.** $C_p^{cell}(B; H_q(F)) = H_p(B^p, B^{p-1}; H_q(F))$ defn.

p-dim cells of B ⟶

$$= \bigoplus_{\sigma} H_p(\sigma, \partial\sigma; H_q(F))$$

$$= \bigoplus H_p(\sigma / \partial\sigma; H_q(F))$$

$$= \bigoplus H_q(F).$$

□

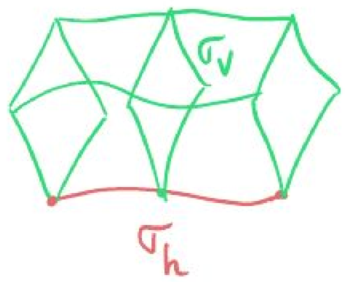
We now have $E^1$. Serre's theorem follows.

# THE SERRE S.S. VIA CUBES

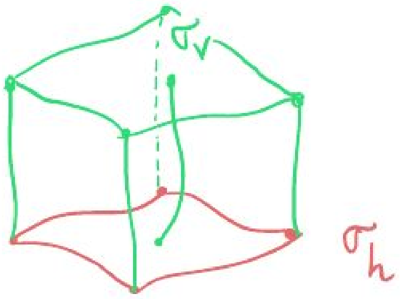
Let $C_*(E)$ be the cubical singular chain complex.

$F_p C_{p+q}(E)$ = span of the singular cubes $\sigma : I^{p+q} \longrightarrow E$ s.t. $\pi \circ \sigma$ is indep. of the last $q$ coords.

Such a cube gives a horizontal $p$-cube $\sigma_h$ and, by restricting to the center of $\sigma_h$, a vertical $q$-cube $\sigma_v$:



$F_1 C_3$



$F_2 C_3$

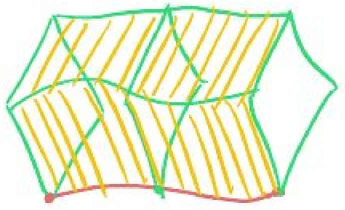
$$\rightsquigarrow \quad F_p C_{p+q}(E) \longrightarrow \bigoplus_{\sigma_h : I^p \to B} C_q(E_{\mathrm{center}(\sigma_h)})$$

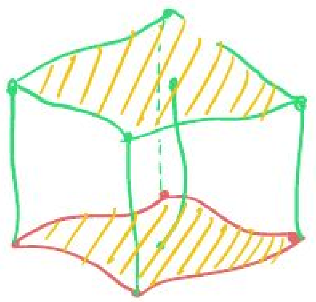
$$\sigma \longmapsto (\sigma_h, \sigma_v)$$

We then mod out by degenerate $\sigma_h$, the ones indep of the last coordinate, and obtain

$$\Phi_0 : E^0_{pq} = G_p C_{p+q}(E) \longrightarrow \bigoplus_{\substack{\sigma_h : I^p \to B \\ \text{nondeg.}}} C_q(E_{\mathrm{center}(\sigma_h)})$$

The differential $\partial_0$ only considers the vertical boundary, i.e. faces obtained by forgetting one of the last q coords:





So if $\Phi_0(\sigma) = (\sigma_h, \sigma_v)$ then:

$$\Phi_0(\partial\sigma) = (-1)^q(\sigma_h, \partial\sigma_v)$$

ie. fiberwise boundary.

So $\Phi_0$ induces a map on homology:

$$\Phi_1 : E^1_{pq} \longrightarrow \bigoplus_{\substack{\sigma_h : I^p \to B \\ \text{nondegen}}} H_q(E_{\text{center}(\sigma_h)}) = C_p(B; \{H_q(E_x)\})$$

Homotopy lifting property for cubes $\Rightarrow \Phi_1$ has an inverse (given $(\sigma_h, \sigma_v)$, homotope it around to get the original $\sigma$).

$\partial_1$ is the horizontal boundary. Need to use parallel transport to show this agrees with the differential on $C_p(B; \{H_q(E_x)\})$

$$\Rightarrow E^2_{pq} = H_p(B; \{H_q(E_x)\}).$$

$\square$

# OTHER SPECTRAL SEQUENCES

**Lyndon-Hochschild-Serre:** Given $1 \to K \to G \to Q \to 1$ there is a spectral sequence with

$$E^2_{pq} = H_p(Q; \{H_q(K)\}) \Rightarrow H_{p+q}(G)$$

**Cartan-Leray:** Given $G \circlearrowright X$ free and proper

$$E^2_{pq} = H_p(G; H_q(X)) \Rightarrow H_{p+q}(X/G)$$

**Or:** $G \circlearrowright X$ cellularly & w/o rotations, $X \simeq *$

$$E^1_{pq} = \begin{cases} \bigoplus_{\sigma \in X_p} H_q(G_\sigma) & p,q \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow H_{p+q}(G)$$

$X_p = \{p\text{-cells}\}$, $G_\sigma$ = stabilizer of $\sigma$.

... and many more (a spectral sequence for every occasion).