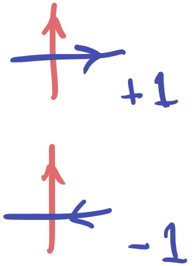
Cup products

$\varphi \in H^k(X)$

$\psi \in H^\ell(X)$

$\varphi \cup \psi \in H^{k+\ell}(X)$



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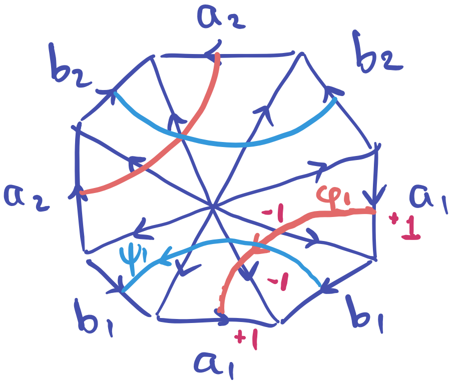


M_2

$\varphi \cup \psi([v_0, \ldots, v_{k+\ell}])$

$= \varphi([v_0, \ldots, v_k]) \cdot \psi([v_k, \ldots, v_{k+\ell}])$

Example: Orientable Surfaces

$X = M_g$. We'll see:

$\cup \longleftrightarrow \hat{i}$

cup   algebraic int.

a_i, b_i form basis for $H_1(M_g)$

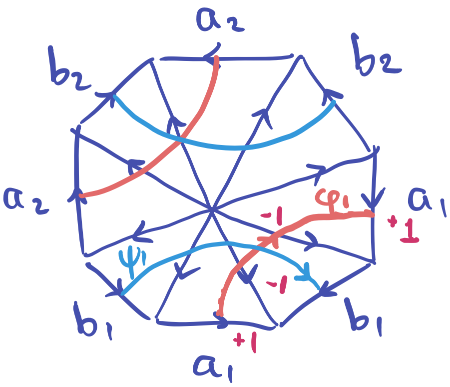
UCT $\Rightarrow H^1(M_g) \cong \mathrm{Hom}(H_1(M_g), \mathbb{Z})$

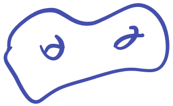
Basis for $H_1 \rightsquigarrow$ basis for $H^1(M_g)$

$a_i \longleftrightarrow \varphi_i$

$b_i \longleftrightarrow \psi_i$

$$\varphi_i(a_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$





$\varphi_1 \cup \psi_1 \in H^2(M_g)$

6 of the Δ's only see one of the two so $\varphi_1 \cup \psi = 0$.

Southern Δ: $0 \cdot 0 = 0$.

SE Δ: $+1 \cdot +1 = 1$

Notice: φ_1, ψ_1 intersect in SE triangle.

We know $H_2(M_g) = \mathbb{Z}$

UCT + H_1 free $\Rightarrow H^2(M_g) \cong \mathrm{Hom}(H_2(M_g), \mathbb{Z}) \cong \mathbb{Z}$

Which elt of $H^2(M_g) \cong \mathbb{Z}$ is $\varphi_1 \cup \psi_1$?

By above: $(\varphi_1 \cup \psi_1)([M_g]) = 1$

↑ fund. class all the Δ's with $+1$

This tells us:

$\varphi_1 \cup \psi_1 \neq 0$ in H^2

$[M_g] \neq 0$ in H_2

& both generate H^2 & H_2

Nonorientable surfaces: See book or 2012 notes.

Naturality

$\alpha, \beta \in H^*(Y)$

$f: X \to Y$

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

already true on cochain level.

Also: cup product for relative cohomology.

The Cohomology Ring

Define $H^*(X;R) = \bigoplus_k H^k(X;R)$

Elts are finite sums.

$\cup$ induces ring structure.

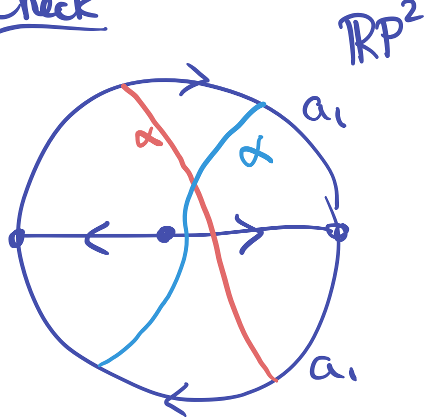
write xy for $x \cup y$.

Claim: $H^*(\mathbb{RP}^2; \mathbb{Z}/2) = \{a_0 + a_1\alpha + a_2\alpha^2 : a_i \in \mathbb{Z}/2\}$

$= \mathbb{Z}/2[\alpha] / \langle \alpha^3 \rangle$

$H^1(\mathbb{RP}^2; \mathbb{Z}) = \langle \alpha \rangle$.

Check



$H^0(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2 = 1$

$H^1(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha \rangle$

$H^2(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha^2 \rangle$

proven below.

What is $\alpha \cup \alpha$?

$\alpha \cup \alpha \text{ (top } \Delta) = 0 \cdot 1 = 0$

$\alpha \cup \alpha \text{ (bot } \Delta) = 1 \cdot 1 = 1$

Check:

$(\alpha \cup \alpha)([\mathbb{R}P^2]) = 1$

$\Rightarrow$ $\alpha \cup \alpha = \alpha^2$ is the generator for $H^2(\mathbb{R}P^2)$