The Cohomology Ring

Apr 4

Last time:

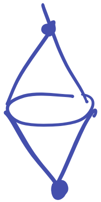
$$H^*(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^3)$$

α is the nonzero elt of $H^1(\mathbb{RP}^2;\mathbb{Z}/2)$

The deg of a ~~poly.~~ monomial. tells you the deg of the corr. elt. of H^*.

"graded ring" $R = \bigoplus_d R_d$

$$R_p \times R_q \subset R_{p+q}$$



One can also show:

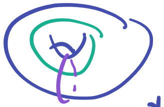
$$H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1}) \qquad |\alpha|=1$$

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]$$

$$H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha] \qquad |\alpha|=2$$

There are spaces with same H_k & H^k gps $\forall k$ but different H^* rings:

$S^1 \vee S^1 \vee S^2 \qquad T^2$



There are distinct spaces with identical H^*:

$$H^*(S^3 \vee S^5) \cong H^*(S(\mathbb{CP}^2)) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(5)}$$

degrees.

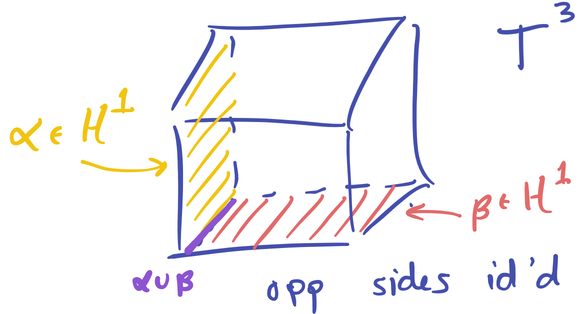
A fact: $\alpha \in H^k$, $\beta \in H^\ell$

Prop. $\alpha \cup \beta = (-1)^{k+\ell}(\beta \cup \alpha)$

if R commutative.

Next goal. $H^*(T^n)$

$T^n = S^1 \times \cdots \times S^1$



We'll show:

$H^k(T^n;\mathbb{Z})$ = free abel. gp with basis

$\alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$ $\quad i_1 < \cdots < i_k$

where $\alpha_i \in H^1(T^n;\mathbb{Z})$ is $p_i^*(\alpha)$ for

α a gen. for $H^1(S^1;\mathbb{Z})$ &

p_i is proj. to i^{th} factor.

To prove this we'll do something more general.

Kunneth formula for $H^*(X \times Y)$.

Tensor products

A, B abelian gps

$A \otimes B$ is the abel. gp gen by

$a \otimes b \qquad a \in A,\ b \in B.$

e.g. $5a_1 \otimes b_1 + 7a_2 \otimes b_2$

relations

$(a + a') \otimes b = a \otimes b + a' \otimes b$

$a \otimes (b + b') =$ similar.

Bilinear maps $A \times B \to C$ same as

homoms $A \otimes B \to C$

Cross product (or, external cup product):

$$H^*(X; \mathbb{Z}) \times H^*(Y; \mathbb{Z}) \to H^*(X \times Y; \mathbb{Z})$$

$$(a, b) \longmapsto p_1^*(\alpha) \cup p_2^*(\beta)$$

bilinear so have homom.

$$H^*(X; \mathbb{Z}) \otimes H^*(Y; \mathbb{Z}) \to H^*(X \times Y; \mathbb{Z})$$

Multiplication on LHS:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$$

Thm (Künneth formula). This is an isomorphism *if* $H^*(X;\mathbb{Z})$ or $H^*(Y;\mathbb{Z})$ are fin. gen & free in each degree.

Exterior algebras

$\Lambda[\alpha_1, \alpha_2, \dots, \alpha_n]$

As a gp: gen by

$$\alpha_{i_1} \cdots \alpha_{i_k}, \quad i_1 < \dots < i_k$$

Multiplication: $\alpha_i \alpha_j = -\alpha_j \alpha_i$.

A, B gps

$A \wedge B$ gen by $a \wedge b$ with all $\otimes$ relations and

$a \wedge b = -b \wedge a$.

in particular: $\alpha_i^2 = 0$.

Cor $H^*(T^n;\mathbb{Z}) \cong \Lambda[\alpha_1, \dots, \alpha_n]$

$|\alpha_i| = 1$

Example of bilinear map: dot product

$(5v + 7w) \cdot u = 5v \cdot u + 7w \cdot u$

$f : \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\text{dot}} \mathbb{R}$

$f(u+v, w) = f(u,w) + f(v,w)$

$\rightsquigarrow \quad \mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}$