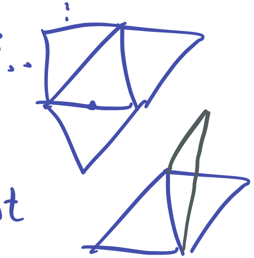
Poincaré Duality

For M a compact orientable n-manifold

$$H_k(M) \cong H^{n-k}(M)$$

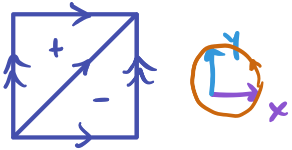
compact: finitely many n-simplices. $\sigma_1, \dots, \sigma_N$

manifold:



not

e.g.



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Or by UCT, modulo torsion we have

$$H_k(M) \cong H_{n-k}(M)$$

Examples

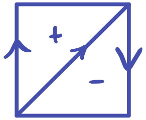
① $H_*(S^n)$ $\quad \mathbb{Z}, 0, \dots, 0, \mathbb{Z}$

② $H_*(M_g)$ $\quad \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$

③ $H_*(T^n)$ $\quad \mathbb{Z}^{\binom{n}{0}} \; \mathbb{Z}^{\binom{n}{1}} \dots \mathbb{Z}^{\binom{n}{n-1}} \; \mathbb{Z}^{\binom{n}{n}}$

The statement of PD gives an explicit $\cong$.

Orientable: $\exists \; \varepsilon_1, \dots, \varepsilon_N \in \{\pm 1\}$ s.t. $[M] = \sum_{i=1}^{N} \varepsilon_i \sigma_i$ is a cycle



The idea of PD

For manifolds:

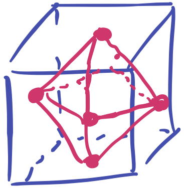
cell structures $\longleftrightarrow$ dual cell structures.

k-cells $\longleftrightarrow$ $(n-k)$-cells

face relations reversed.

examples

(1)



dual cell structures on S^2.

& other platonic solids

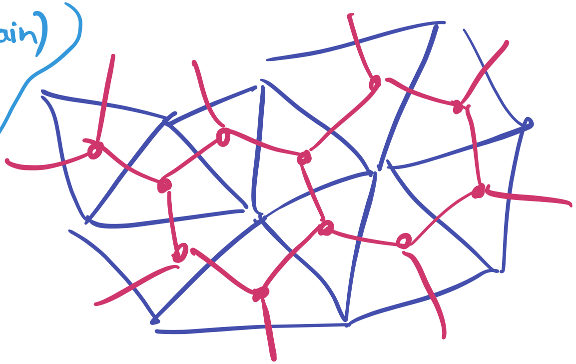
A pink object (chain) is a cocycle if when pair with any blue ∂, get 0

$\Updownarrow$

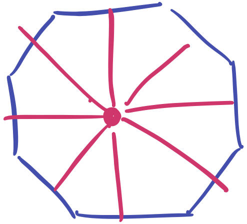
pink cycles

$\Downarrow$

PD.



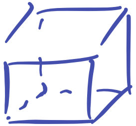
(2)



M_g

dual cell str. $\cong$ orig. cell str.

(3) T^n



ditto

Duality with $\mathbb{Z}/2$ coeffs

Can ignore signs. $\rightsquigarrow$ natural pairing between cell str C & dual C^*

$$C_i \longleftrightarrow C^*_{n-i}$$

Under this identification

$$\partial : C_i \longrightarrow C_{i-1}$$
$$\sigma \longmapsto \text{sum of faces.}$$

becomes

$$\delta : C^*_{n-i} \longrightarrow C^*_{n-i+1}$$
$$\sigma^* \longmapsto \text{sum of dual cells of which } \sigma^* \text{ is a face.}$$

$$\rightsquigarrow H_i(C;\mathbb{Z}/2) \cong H^{n-i}(C^*;\mathbb{Z}/2).$$
$$\begin{array}{ccc} \cong & & \cong \\ H_i(M;\mathbb{Z}/2) & & H^{n-i}(M;\mathbb{Z}/2) \end{array}$$

This proves PD for $\mathbb{Z}/2$ coeffs.

Cap product

$k \geq \ell$

$$\frown : C_k(X) \times C^\ell(X;\mathbb{Z}) \longrightarrow C_{k-\ell}(X)$$

$$(\sigma, \varphi) \longmapsto \underbrace{\varphi(\sigma|_{[v_0,\dots,v_\ell]})}_{\text{number}} \sigma|_{[v_\ell,\dots,v_k]}$$

As usual, need to check this induces a map on co/homology. The required formula is:

$$\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

$\rightsquigarrow$ cycle $\frown$ cocycle = cycle

cycle $\frown$ coboundary = boundary

boundary $\frown$ cocycle = boundary

Two facts:

- linear in each var
- natural

$f : X \to Y$

$$f_*(\sigma) \frown \varphi = f_*(\sigma \frown f^*(\varphi))$$

in $H_*(Y)$

Thm (PD)

M = compact n-manifold with orientation $[M]$. Then

$$H^k(M) \longrightarrow H_{n-k}(M)$$

$$\varphi \longmapsto [M] \frown \varphi$$

is an $\cong$.