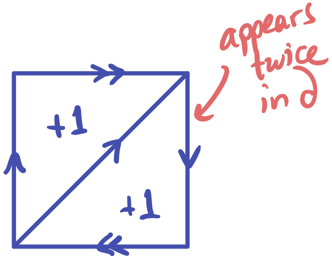
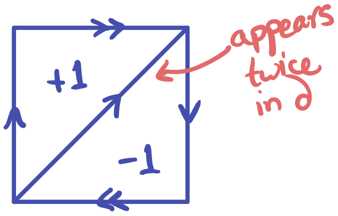
Poincaré Duality

M compact, orientable n-manifold.

$$H^{n-k}(M) \longrightarrow H_k(M)$$
$$\varphi \longmapsto \varphi \cap [M]$$

makes sense with $\mathbb{Z}$-coeff exactly when M is orientable.





$\Rightarrow$ not orientable

Orientable: can put ± 1 on each simplex of M and get an n-cycle.

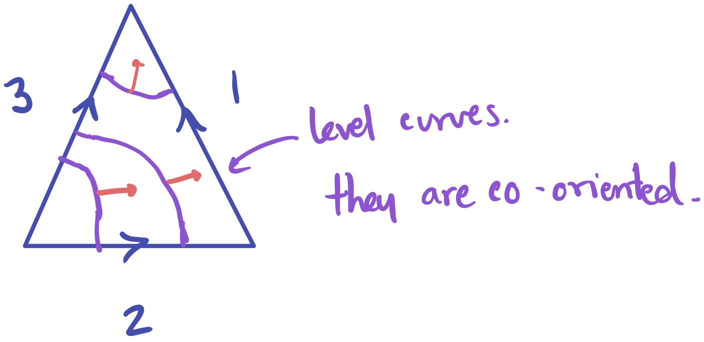
Note: every manifold is $\mathbb{Z}/2$-orientable. If you put $(+)1$ on each n-simplex, get a $\mathbb{Z}/2$ cycle.

φ (n-k cocycle) $\cap$ $[M]$ (n cycle) is a k-cycle,

on each simplex σ of M, evaluate φ on "front" n-k simplex of σ $\rightsquigarrow$ number

Scale the "back" k simplex of σ by that number.

We know: cocycles $\rightsquigarrow$ dual objects



Recall: $\varphi \in H^k$

$\psi \in H^\ell$

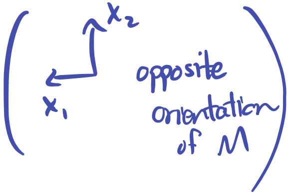
$\varphi \smile \psi \in H^{k+\ell}$

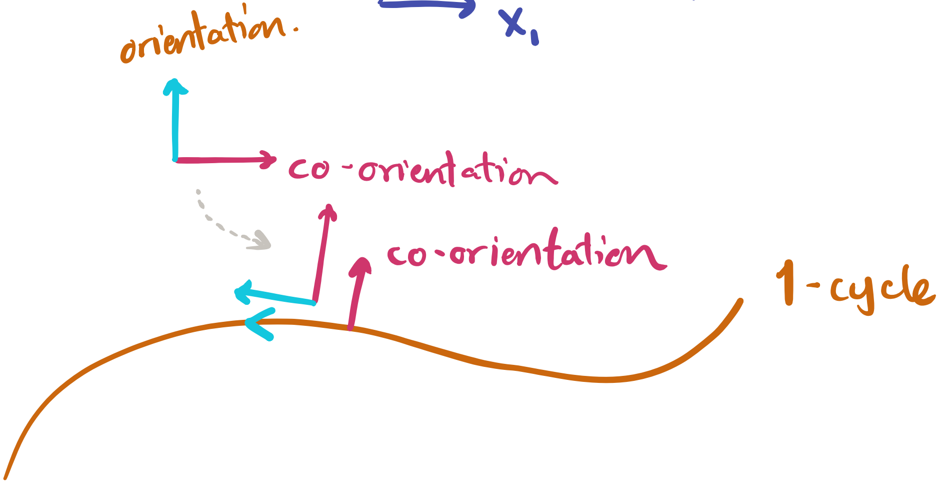
Two claims: ① Cup product of cocycles is intersection of dual objects.

② Cap product is "pushing" or homotoping the dual objects

In an orientable manifold:

co-orientations $\longleftrightarrow$ orientations





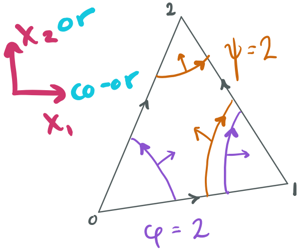
Cup & Cap notes from my Teaching page

Apr 8

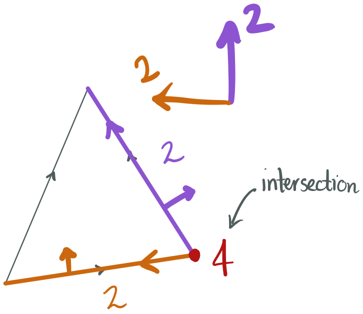
Cup

Idea. To find $\varphi \cup \psi$, push $\overset{\text{dual of}}{\wedge}$ φ up, push $\overset{\text{dual of}}{\wedge}$ ψ down and intersect

Example 1. $n=2$, $k=1$



push



$\varphi, \psi \in H^1$

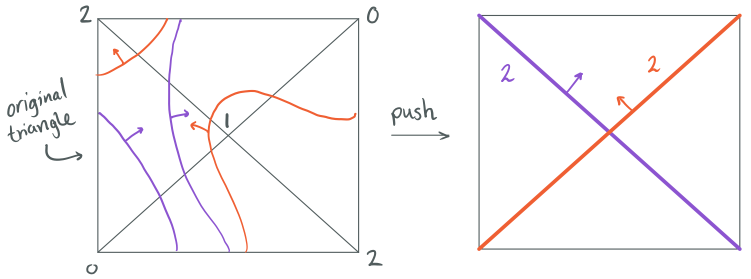
$\varphi \cup \psi \in H^2$

Using def of $\cup$:

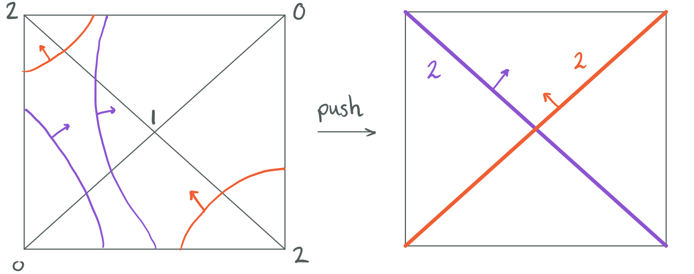
$\varphi \cup \psi(\sigma) = 2 \cdot 2$

$= 4$

Can view same example in context of nearby triangles:



We can modify the curves by homotopy, giving cohomologous cochains:



Cup, Cap, and Poincaré Duality

Poincaré duality. $H^k(X) \xrightarrow{\cong} H_{n-k}(X)$
$\varphi \longmapsto [M] \frown \varphi$

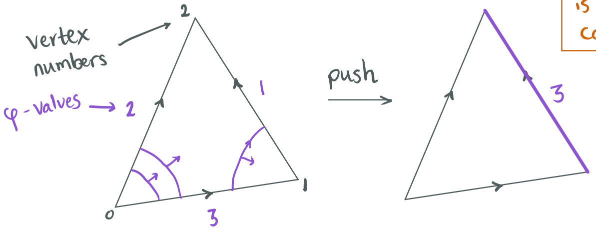
Also. Under this isomorphism, cup product corresponds to intersection: $\varphi \cup \psi \longmapsto \varphi^* \cap \psi^*$

We'll work with Δ-complexes, simplicial (co)homology.

Cap

Idea. Realize cohomology class φ as "intersect with dual object." Push dual in each simplex toward highest vertex (this is well-defined across different simplices in a Δ-complex).
Result is $[M] \frown \varphi = \varphi^*$

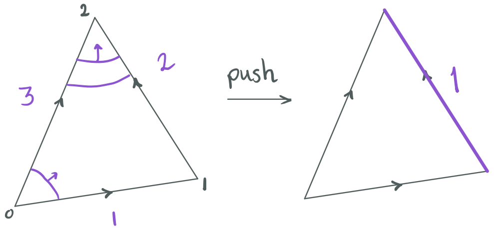
Example 1. $n=2,\ k=1$



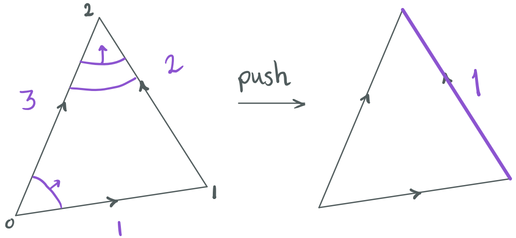
This is exactly what $\varphi \frown [M]$ gives!

Note: in a manifold, orientation is same as co-orientation

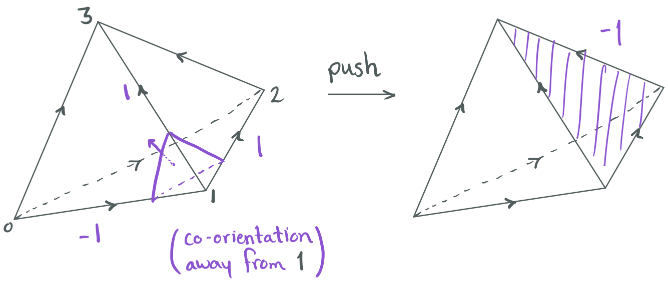
Example 2. $n=2,\ k=1$



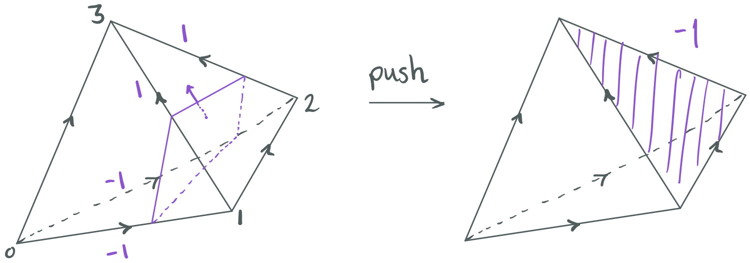
Example 2. $n=2,\ k=1$



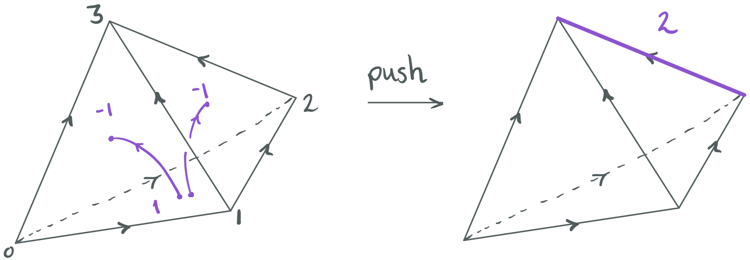
Example 3 $n=3,\ k=1$



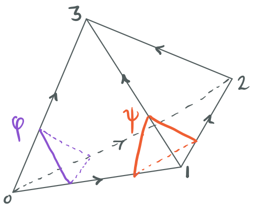
Example 4 $n=3,\ k=1$



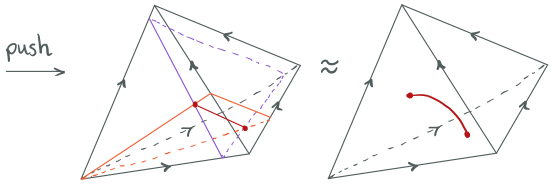
Example 5 $n=3,\ k=2$



Example 2. $n=3$, $k=1,1$ (mod 2 this time)



Have $\varphi \cup \psi \in H^2 \rightsquigarrow$ should be dual to a 1-cell. If we push all the way and intersect, get a point (not what we want). If we push almost all the way, we get what we want:



This is dual to $\varphi \cup \psi$!

Note: In the earlier examples, pushing almost all the way also works.

Claim. This proves P.D.