Apr 11

Applications of PD

PD: modulo torsion, have.

$$H^k(M) \cong H_{n-k}(M)$$

$$\overset{\text{UCT}}{\cong} H_k(M)$$

① Euler characteristic.

Prop. dim M odd $\Rightarrow$

$$\chi(M) = 0.$$

Pf. $\chi(M) \cong \sum_{k=0}^{n} (-1)^k \operatorname{rk} H_k(M;\mathbb{Z})$

Apply PD $\square$

e.g. $\mathbb{Z} \quad 0 \quad 0 \quad \mathbb{Z}$

rk: $1 \quad 0 \quad 0 \quad 1$

$\rightsquigarrow +1 - 0 + 0 - 1 = 0.$

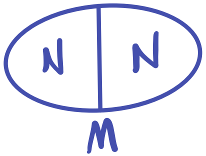
Prop. dim M even.

$\chi(M)$ odd

$\Rightarrow M$ is not a boundary of $(n+1)$-man.

e.g. $M = \mathbb{RP}^2$.

Pf. Suppose $M = \partial N$

$2N$: 

$0 = \chi(2N) = 2\chi(N) - \chi(M).$ cont. $\square$

② Intersection Forms

F = field. e.g. $\mathbb{Q}$.

$H_k(M;\mathbb{Q}) = H_k(M;\mathbb{Z})/\text{torsion} \otimes \mathbb{Q}$.

Have:

or: intersect

$$\langle\ ,\ \rangle : H_k(M) \times H_{n-k}(M) \xrightarrow{\cup^*} F$$

"intersection form"

Prop. Int. form is nonsingular.

i.e. $\forall\ \alpha \neq 0$, $\langle \alpha, \cdot \rangle \neq 0$.

i.e. $\alpha \neq 0$ in $\mathrm{Hom}(H_{n-k}, F)$.

Pf. $H_k(M) \xrightarrow{PD} H^{n-k}(M) \xrightarrow{UCT} \mathrm{Hom}(H_{n-k}, F)$

$\xrightarrow{PD} \mathrm{Hom}(H_k, F)$. $\square$

These are all $\cong$

Now say $\dim M = n = 2k$ even.

$\rightsquigarrow$ int. form on $H_k(M)$.

$\rightsquigarrow H_k(M) \times H_k(M) \longrightarrow \mathbb{Z}$

tors. free part of $\mathbb{Z}$ homol.

$\rightsquigarrow \mathbb{Z}$-matrix $\langle \alpha_i, \alpha_j \rangle$

Prop. This matrix/int. form is unimodular, i.e. det 1.

Pf. Let $\alpha_1, \dots, \alpha_\ell$ basis.

$\beta_1, \dots, \beta_\ell$ dual basis (by prev. Prop)

$\rightsquigarrow \langle \alpha_i, \beta_j \rangle = \delta_{ij}$

Change of basis has det 1 $\square$

If $n = 4k$, int. pairing is symmetric.

$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$$

$\Longrightarrow$ eigenvals real.

b_{2k}^{+}, b_{2k}^{-} # of pos, neg eigenvals.

$$\sigma(M) = b_{2k}^{+} - b_{2k}^{-}$$

"signature"

Let Ω_d^{SO} = cobordism group of d-dim. compact oriented manifolds

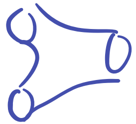
$M \sim N$ if

$M - N = \partial W$

addition: disjoint union.

id: $\emptyset$.

$\Omega_1^{SO} = 0$



Thm. $\Omega_d^{SO} = 0$ $d \le 3$ (easyish for $d \le 2$)

Thm (Thom). In dim 4, σ is a cobordism invt. $\Rightarrow$ Ω_4^{SO} has $\mathbb{Z}$-many elts.

$\rightsquigarrow$ Fields medal

Alexander duality

Thm. K compact, locally compact, nonempty subspace of S^n

$\Rightarrow$ $\tilde{H}_i(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$

$\forall i$.

Surprisingly: LHS does not depend on the embedding.

application. $H_1(S^3 \setminus \text{knot}) \cong \mathbb{Z}$.

Gordon-Luecke: $\pi_1(S^3 \setminus \text{knot})$ determines the knot.

Cor. $X \subseteq \mathbb{R}^n$ compact, loc comp.

$\Rightarrow H_i(X; \mathbb{Z}) = 0 \quad i \geq n$

$= \text{torsion free}$

$i = n-1, n-2$.

Pf. $\tilde{H}^{-k} = 0$

$\tilde{H}^0, \tilde{H}^1$ always torsion free

Application. Klein bottle does not embed in $\mathbb{R}^3$ or S^3.

$H^2(K.B.)$ has torsion.