Spectral Sequences

The LES for a pair gives $H_*(X)$ in terms $H_*(X, A)$, $H_*(A)$

Similar LES for triple.

Not for quadruples.

Answer: Spectral seqs.

Filtration

Apr 13

X = CW complex

Filter by subcomplexes

$X_0 \subseteq X_1 \subseteq \cdots$

k-chains in X_p = "

$\rightsquigarrow$ filtration of $C_*(X)$: $F_p C_k$

$\rightsquigarrow$ assoc. graded modules:

$$G_p C_k = F_p C_k / F_{p-1}(C_k)$$

"new k-chains at stage p"

Examples ① $X_i = X^{(i)}$ i-skeleton.

② For a fiber bundle

X_i = preimage of i-skeleton of base.

e.g. product.

or: Möbius band.

Other examples of fiber bundles:

$SO(n)$ $\quad SO_2 \to SO_3 \downarrow S^2$ image of north pole.

$\Rightarrow$ can understand $H_*SO(n)$ by induction & Spec Seq.

Recall: $G_pC_k = F_pC_k / F_{p-1}(C_k)$

Filtered Chain complexes

We have $\partial F_pC_k \subseteq F_pC_{k-1}$

$\rightsquigarrow$ induced $\partial : G_pC_k \to G_pC_{k-1}$

$\rightsquigarrow$ associated graded chain complex

(G_pC_*, ∂)

Hope. $H_*(G_pC_*)$ easy-ish to compute, understand $H_*(X)$ as a limit...

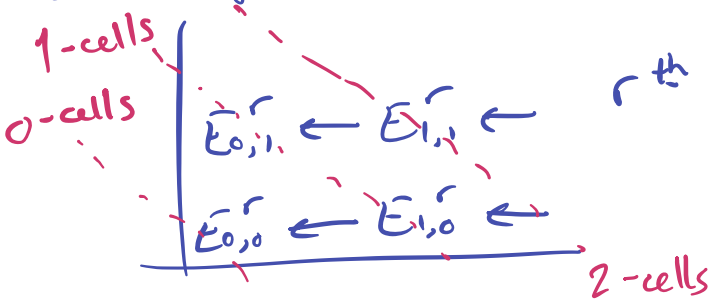
Another idea of what a spectral seq is

A spectral seq has pages.

Each page is a 2D grid of vect spaces. (first quadrant)
(if we work over a field)

There are differentials, and we get from one page to the next by taking homology. (maps)

Each page looks like



r^{th} page

Taking homology turns the page

Arrows/maps change direction

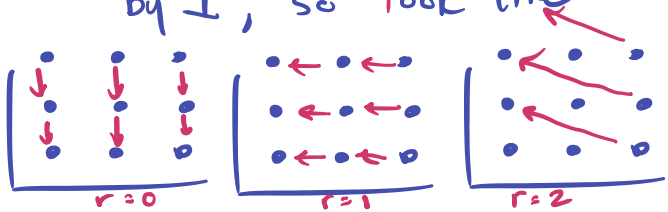
Keep doing this forever.

The $E^r_{p,q}$ will correspond to

k-chains where $p+q=k$.

e.g. $E^0_{p,q} = G_p C_{p+q}$

The differentials reduce the dim by 1, so look like



= ker/im.

In favorable cases, each $E^r_{p,q}$ stabilizes with r.

$\rightsquigarrow E^\infty_{p,q}$ is this term.

$H_k(X)$ given by $E^\infty_{p,q}$ with $p+q=k$.

Think about paintball:

Basis elts of the $E_{p,q}$ are players.

All get one paintball.

If you shoot a nontrivial player, both you & the target are eliminated.

If you shoot trivial player, get to stay for next rand.

Sometimes a spectral seq degenerates, meaning all terms stabilize at the same time.

Using spectral sequences

In LES, hope for 0's.

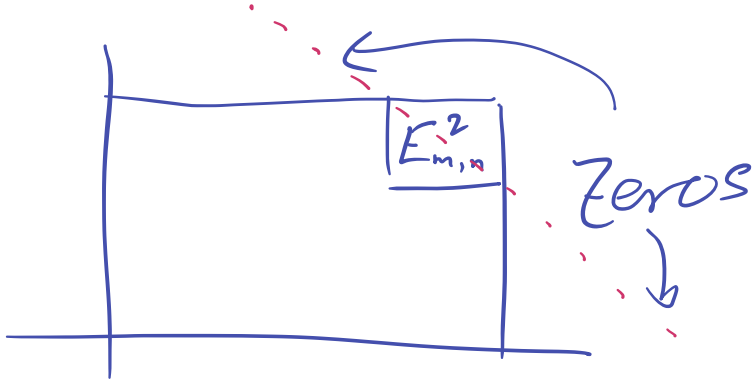
Same here.

For example, in Serre SS for fiber bundles

$E^2_{p,q} = H_p(B, H_q(F))$

If B is m-dim & F is n dim.

then.



$\Rightarrow E^2_{m,n} = E^\infty_{m,n} = H_{m+n}(X)$

this is a good excuse for the indexing.