Spectral Sequences

Apr 15

$X_0 \subseteq X_1 \subseteq \cdots$

$\bigcup X_i = X$ CW complex

F_pC_k k-chains in X_p

$G_pC_k = F_pC_k / F_{p-1}C_k$

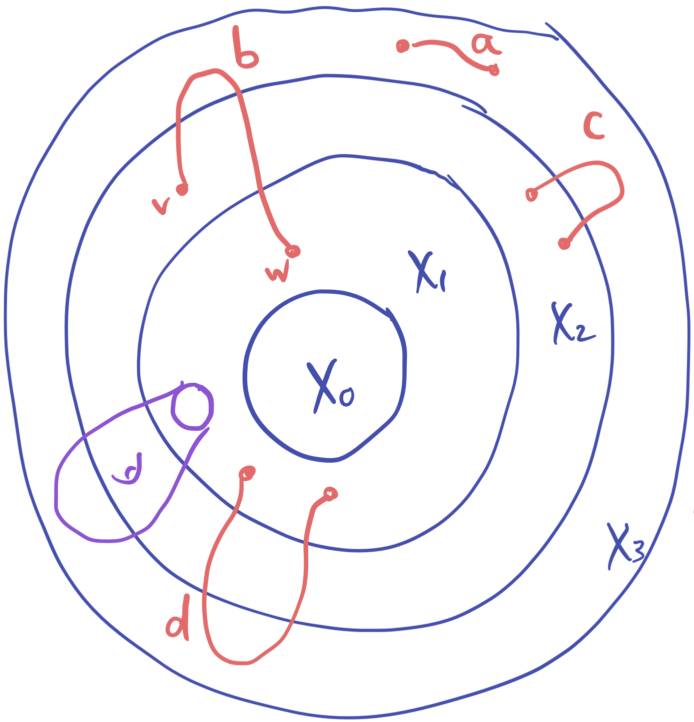
$\partial_p : G_pC_k \longrightarrow G_pC_{k-1}$

In cartoon:

$\partial_3 a \neq 0 \quad \partial_3 c = 0$

$\partial_3 b = 0 \quad \partial_3 d = 0.$

Cartoon



$\langle v, w\rangle / v \cong \langle v, w\rangle / \langle v - w\rangle$

More defns

F_pC_k k-chains in X_p

$G_pC_k = F_pC_k / F_{p-1}C_k$

$\partial_p : G_pC_k \longrightarrow G_pC_{k-1}$

$E^0_{p,q} = G_pC_{p+q}$ down arrows

$\partial_0 : E^0_{p,q} \longrightarrow E^0_{p,q-1}$ (usual ∂)

p+q chains p+q-1 chains

$E^1_{p+q} = H_{p+q}(G_pC_*)$

and $\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ ← left arrows.

defined as follows:

given $\alpha \in E^1_{p,q}$, represent it by a chain $x \in F_pC_{p+q} \rightsquigarrow \partial x \in F_pC_{p+q-1}$

$\rightsquigarrow \partial_1(\alpha) = [\partial x]$ ← mod out by F_{p-1}

Exercise: ∂_1 well def & $\partial_1^2 = 0$

Again: $E^2_{p,q}$ obtained by taking homol.

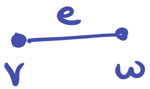
$$E^2_{p,q} = \frac{\ker(\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q})}{\mathrm{im}(\partial_1 : E^1_{p+1,q} \longrightarrow E^1_{p,q})}$$

We get $E^r_{p,q}$ by repeating the process.

Or a closed formula

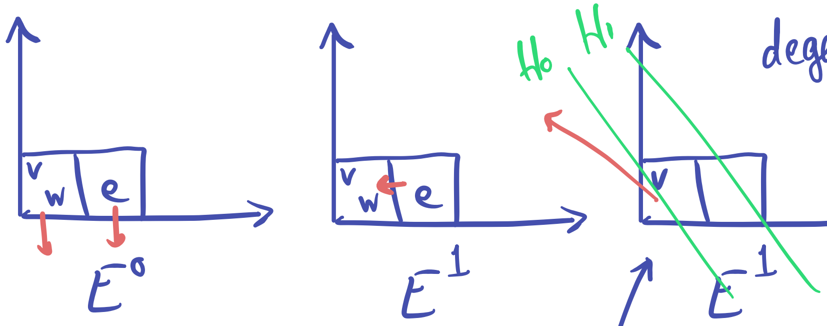
$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} : \partial x \in F_{p-r} C_{p+q-1}\}}{(F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})) \cap \text{numerator}}$$

Baby Example 1

$X =$ 

$X_0 = X^{(0)}$

$X_1 = X^{(1)} = X$



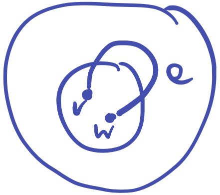
degenerates!

$\Longrightarrow H_0 = \mathbb{Z}$

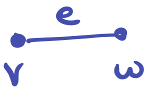
$H_k = 0 \quad k > 0.$

really

$\langle v, w\rangle / \langle v - w\rangle \cong \langle v \rangle$



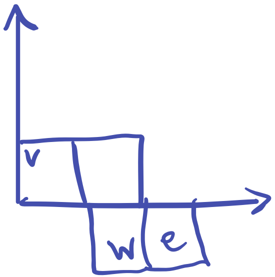
Baby Example 2

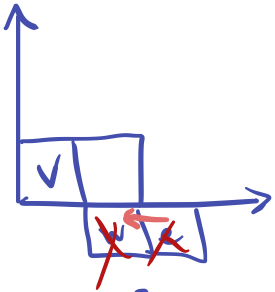
$X =$ 

$X_0 = v$

$X_1 = \{v, w\} = X^{(0)}$

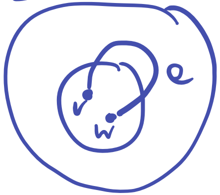
$X_2 = X^{(1)} = X$





degenerates!

$E^0 = E^1$ $\qquad$ E^2



$\Longrightarrow H_0 = \mathbb{Z}$

$H_k = 0 \quad k > 0.$