Spectral Seq's

Apr 18

Goal: $H_*(SU(n))$

$X_0 \subseteq X_1 \subseteq \cdots$

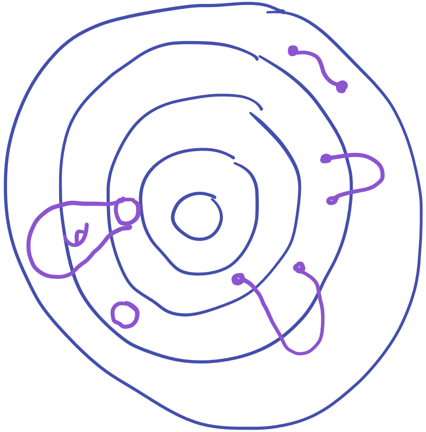
filtration of X.

$F_p C_k$: free abel gp on singular k-chains in X_p

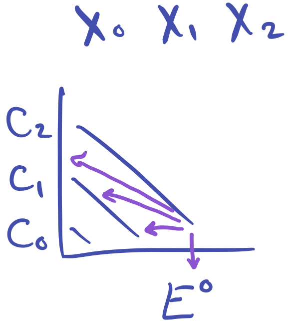
$\rightsquigarrow G_p C_k = F_p C_k / F_{p-1} C_k$

& $\partial_r : G_p C_k \longrightarrow G_{p-r} C_{k-1}$

Cartoon:



$E^0_{p,q} = G_p C_{p+q}$



Toddler Example

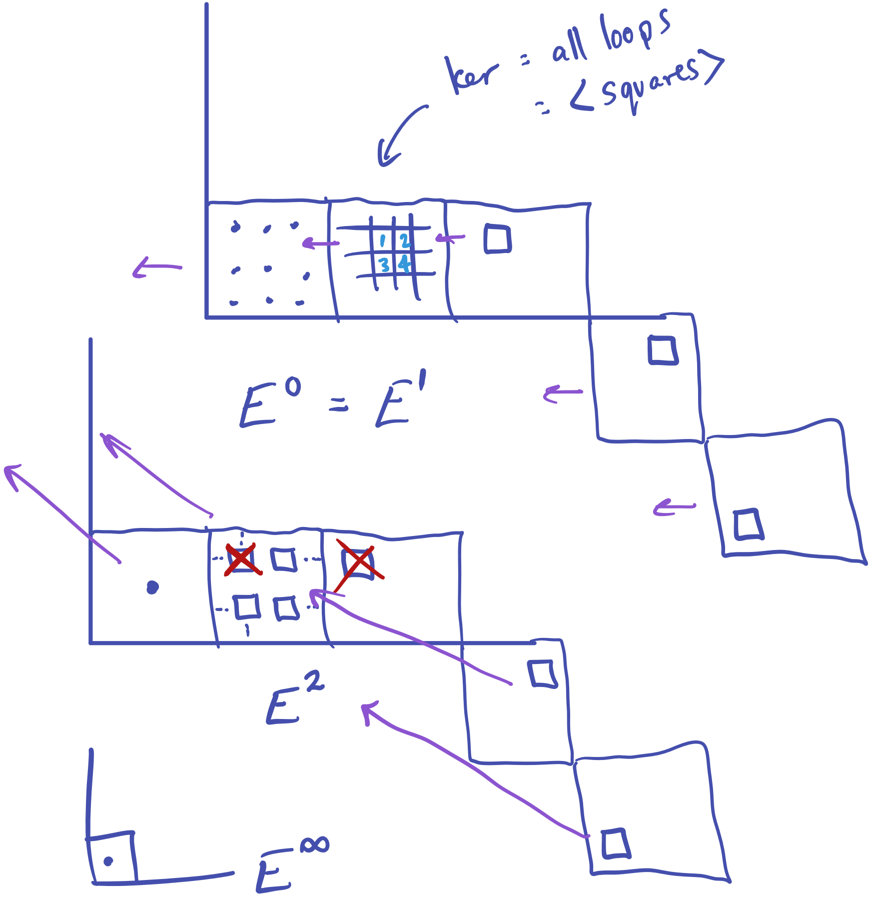
$X = \mathbb{R}^2$ with usual decomp into squares

$X_0 = X^{(0)}$

$X_1 = X^{(1)}$

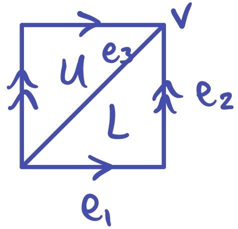
$X_i = X_{i-1} \cup$ one square

$i \geq 2.$

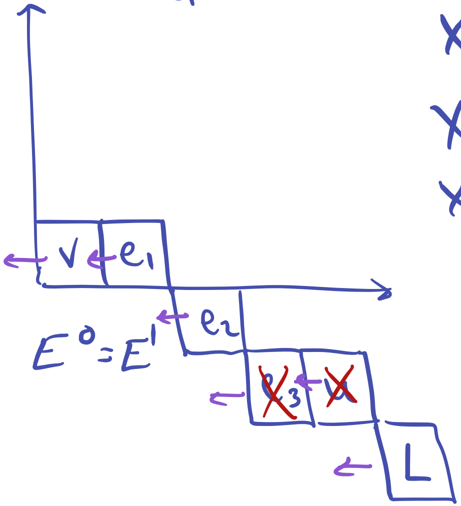


The One-at-a-Time Spectral Seq

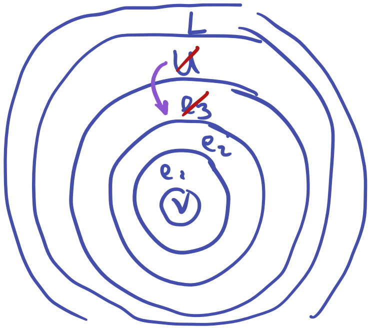
$X =$

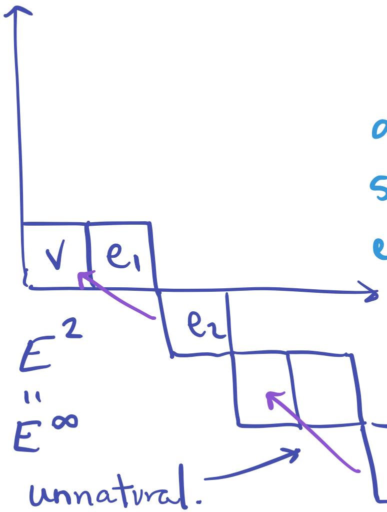


$$X_0 = X^{(0)} = \{v\}$$
$$X_1 = X_0 \cup e_1$$
$$X_2 = X_1 \cup e_2$$
$$X_3 = X_2 \cup e_3$$
$$X_4 = X_3 \cup U$$
$$X_5 = X_4 \cup L$$



$E^0 = E^1$





$E^2 = E^\infty$

unnatural.

add U since U eliminated e_3.

$$\Longrightarrow \quad H_0(T) = \mathbb{Z} = \langle v \rangle$$
$$H_1(T) = \mathbb{Z}^2 = \langle e_1, e_2 \rangle$$
$$H_2(T) = \mathbb{Z} = \langle L+U \rangle$$

Application: Cellular = Singular

Prop. For X = cell complex, $H_*(X) = H_*^{cell}(X)$

Pf. Use: Spec seq. correctly computes sing. hom H_*

Let $X_i = X^{(i)}$

$\rightsquigarrow E^0_{pq} = C^{sing}_{p+q}(X^{(p)}) / C^{sing}_{p+q}(X^{(p-1)})$

$\rightsquigarrow E^1_{pq} = H^{sing}_{p+q}(X^{(p)}, X^{(p-1)})$

$$= \begin{cases} C_p^{cell}(X) & q=0 \\ 0 & q \neq 0 \end{cases}$$

bottom row of E^1

free abel gp on p-cells

Now $(q=0)$

$\partial_1 : H_p(X^{(p)}, X^{(p-1)}) \to H_{p-1}(X^{(p-1)}, X^{(p-2)})$

gluing map.

$\Rightarrow E^2$ page is $H_*^{cell}(X)$ in bottom row

$\Rightarrow E^2 = E^\infty$

The prop. follows. $\square$

$$E^0_{pq} = C_{p+q}(X^{(p)})/C_{p+q}(X^{(p-1)})$$

$$0$$

$$\underline{C_0(X^{(0)})/C_0(X^{(-1)}) \leftarrow C_1(X^{(1)})/C_1(X^{(0)})} \leftarrow C_2(X^{(2)})/C_2(X^{(1)})$$

$$C_0(X^{(0)})/C_0(X^{(-1)}) \leftarrow C_1(X^{(2)})/C_1(X^{(1)})$$

$$\leftarrow C_0(X^{(2)})/C_0(X^{(1)})$$

E^0

$$H_0(X^{(0)}, X^{(-1)}) \leftarrow H_1(X^{(1)}, X^{(0)}) \leftarrow \text{etc} \ldots$$

E^1

$$\Rightarrow E^2 = E^\infty = \text{cell. hom.}$$