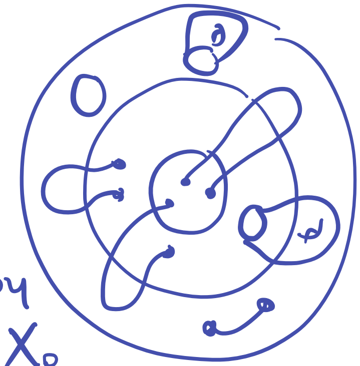
Spectral Sequences

$X_0 \subseteq X_1 \subseteq X_2 \cdots$

$F_p C_{p+q}$ = abel gp. gen by $p+q$ chains in X_p



$$G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$$

Apr 20

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} : \partial x \in F_{p-r} C_{p+q-1}\}}{(F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})) \cap \text{numer.}}$$

= rth approx of cycles / rth approx of boundary.

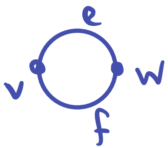
differential ∂ is: choose a rep, take boundary, intersect w/ $F_{p-r} C_*$

Thm. $(F_p C_*)$ = filtered complex, E^r_{pq}, ∂ as above.

- $\partial_r : E^r_{pq} \to E^r_{p-1,q+r-1}$ well def & $\partial_r^2 = 0$.
- E^{r+1} is homology of (E^r, d_r) i.e. $E^{r+1}_{p,q} = \ker d_r / \operatorname{im} d_r$.
- If C_i bdd wrt filtration then $\forall\, p,q\ \exists$ large r s.t. $E^r_{p,q} = G_p H_{p+q} C_*$

Can do w/ sing, cell, simpl. homology.

One At A Time Spectral Seq

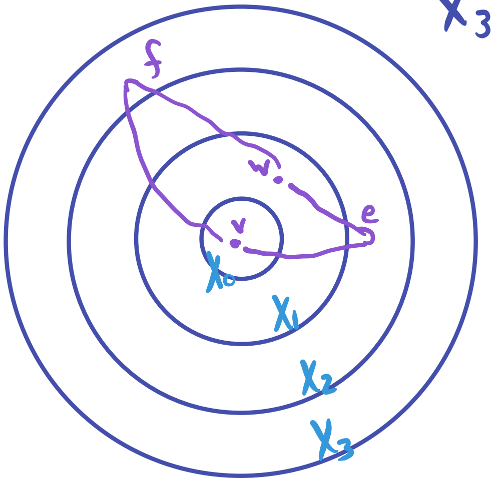
example 

$X_0 = \{v\}$

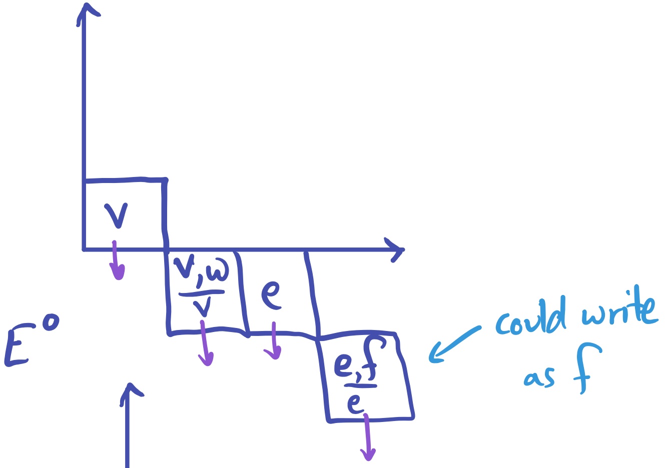
$X_1 = \{v, w\} = X^{(0)}$

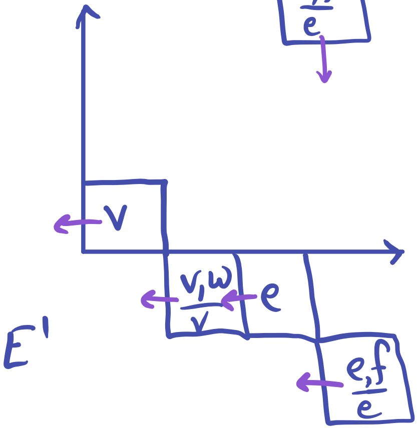
$X_2 = \{v, w, e\}$

$X_3 = \{v, w, e, f\} = X$

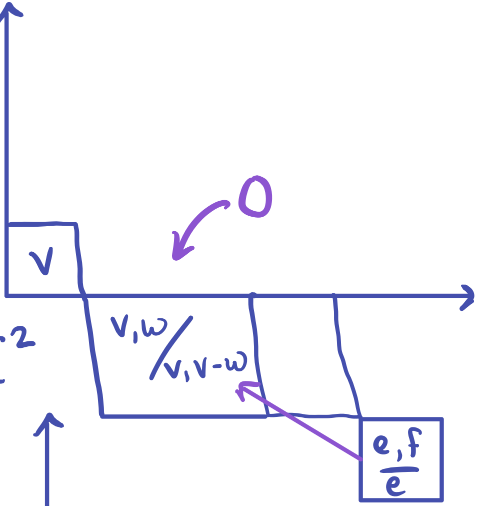


Like the torus example from last time: ∂e is in two different gradings (X_0 & X_1).

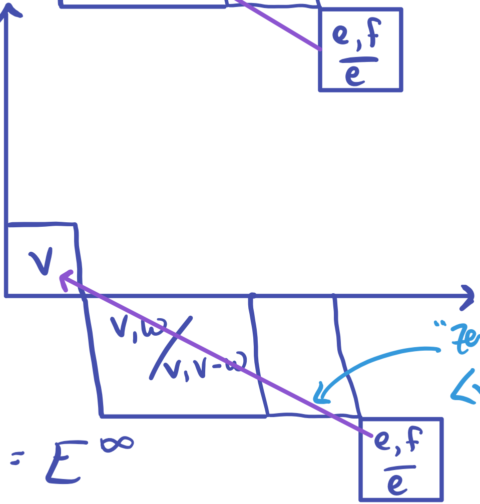




E^2



$E^3 = E^\infty$

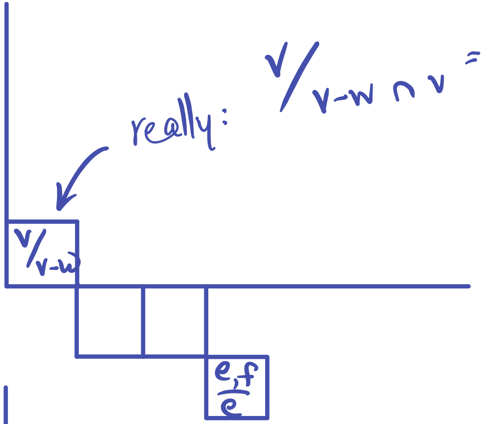


"zero map" since $\langle v-w\rangle \cap \langle v\rangle = 0$

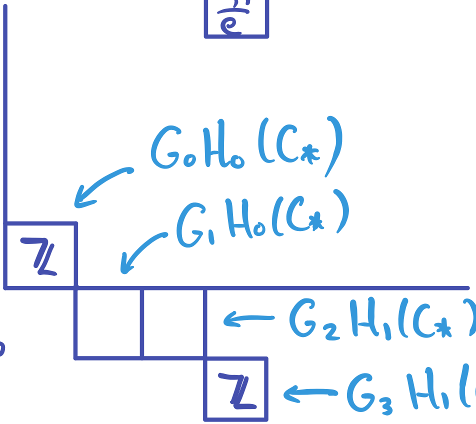
by our defn of ∂_3:
choose rep f,
take $\partial f = v-w$
intersect with F_0C_0 $= 0$.

$E^4 = E^\infty$

really: $V/_{v-w\,\cap\, v} = V/_0$



E^∞



$G_0H_0(C_*)$

$G_1H_0(C_*)$

$G_2H_1(C_*)$

$G_3H_1(C_*)$

$$\Longrightarrow \quad H_0(X) \cong H_1(X) \cong \mathbb{Z}, \quad H_i(X) = 0 \quad i>1$$

Optional HW

Redo torus
example
from last
time