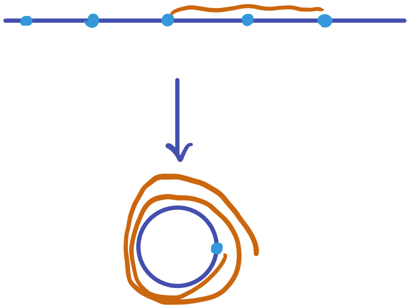
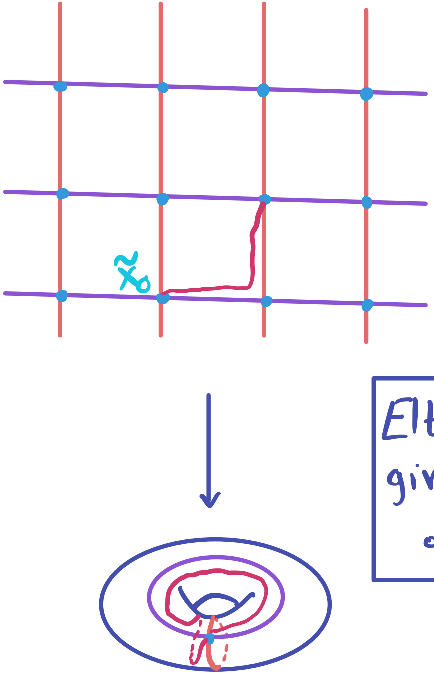
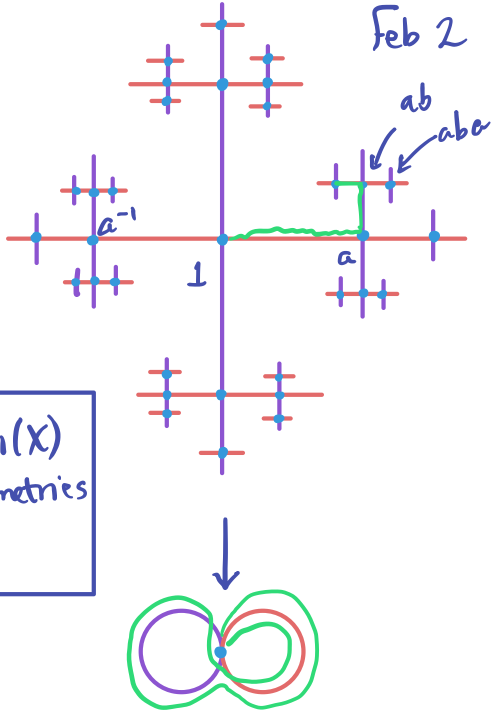
Covering Spaces

Feb 2







Elts of $\pi_1(X)$ give symmetries of $\tilde{X}$

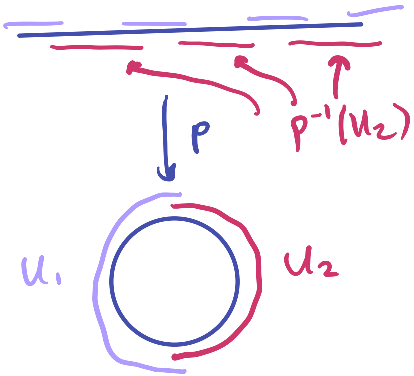
The key to proving $\pi_1(S^1) \cong \mathbb{Z}$ is path/homotopy lifting from S^1 to $\mathbb{R}$. We can do this for other spaces...

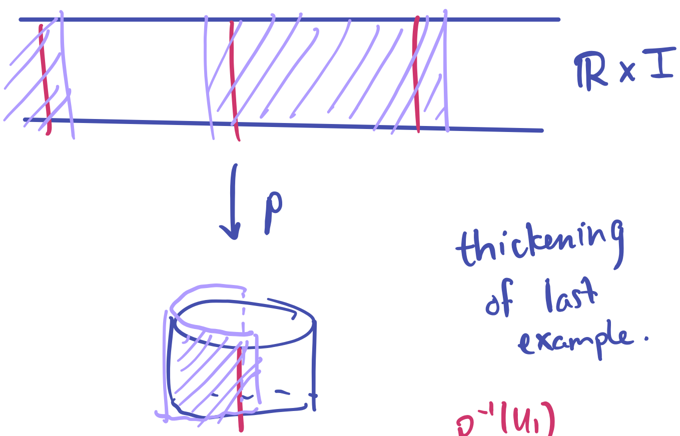
Covering spaces

A cov sp of X is connected $\tilde{X}$ with $p: \tilde{X} \to X$ satisfying:

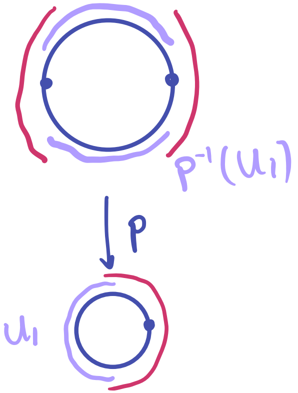
$\exists$ open cover $\{U_\alpha\}$ of X s.t. each $p^{-1}(U_\alpha)$ is a disj. union of open sets, each homeo to U_α.

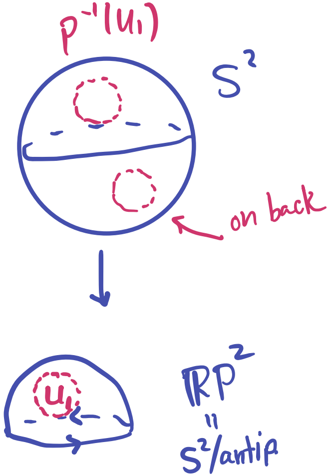
Examples

① 

② 

thickening of last example.

③ 

④ 

A universal covering space is one that is simply connected

examples: $\mathbb{R} \to S^1$ $\quad \mathbb{R}^2 \to T^2$

$S^2 \to \mathbb{RP}^2$ $\quad T_4 \to S^1 \vee S^1$

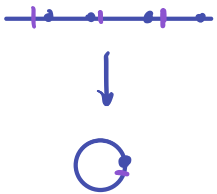
Will show: existence/uniqueness.

We'll also see:

① $\pi_1(X) \longleftrightarrow$ symmetries of univ cover

② subgroups of $\pi_1(X) \longleftrightarrow$ covers of X

example: S^1



① via path lifting, ② via path proj.

Fundamental Theorem

$p: \tilde{X} \to X$ covering map

$G(\tilde{X})$ = deck transformation group

= p-equivariant symmetries of $\tilde{X}$

i.e.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$p \circ f = p$

$H = p_* \pi_1(\tilde{X})$

$N(H)$ = normalizer in $\pi_1(X)$ of H

= largest subgp of $\pi_1(X)$ s.t. $H \trianglelefteq N(H)$

= elts of $\pi_1(X)$ that conj H to itself

If $H \trianglelefteq \pi_1(X)$, $N(H) = \pi_1(X)$.

Thm. $1 \to H \to N(H) \to G(\tilde{X}) \to 1$ is exact. The map $N(H) \to G(\tilde{X})$ is

$\gamma \longmapsto$ unique deck transf. taking $\tilde{x}_0$ to $\tilde{\gamma}(1)$

Exact sequence: Image of each map is kernel of next one.

Short exact seq: $1 \to K \xrightarrow{i} G \xrightarrow{f} Q \to 1$

exactness same as: i inj.

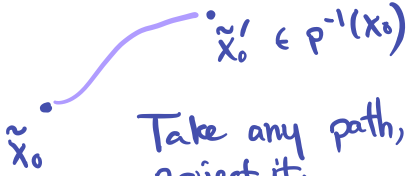
f surj.

Same as: $Q \cong G/K$.

Cor. $H = 1 \Rightarrow \pi_1(X) \cong G(\tilde{X})$

Cor. $H \trianglelefteq \pi_1(X) \Longleftrightarrow G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$

Pf. ($\Rightarrow$)



Take any path, project it.
$\rightsquigarrow$ elt of $\pi_1(X)$

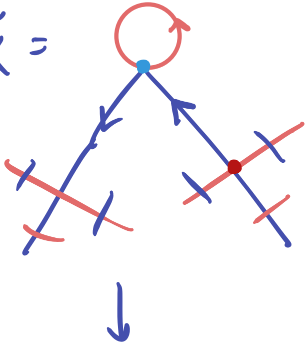
Also: There is a bijection:

$$\left\{\begin{matrix}\text{based cov sp}\\ \text{of } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Subgps}\\ \text{of } \pi_1(X)\end{matrix}\right\}$$

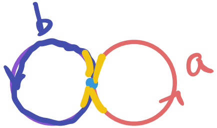
top. gp thy.

Example

$\tilde{X} =$



$X =$



$\pi_1(\tilde{X}) = \mathbb{Z}$

$H = \mathbb{Z} = \langle a \rangle \ntrianglelefteq F_2 .$