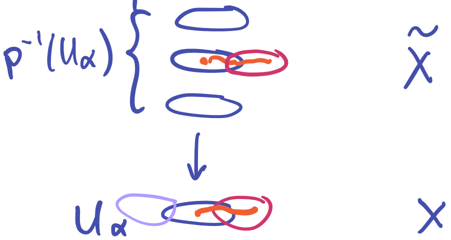
Covering spaces

Feb 4

connected

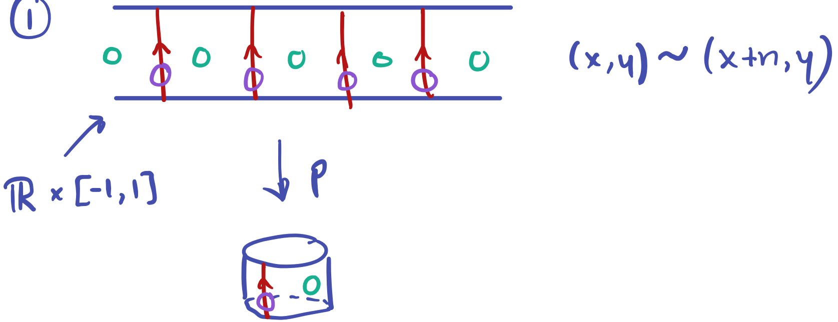
A covering space of X is a connected space $\tilde{X}$ with a map $p: \tilde{X} \to X$ satisfying:

$\exists$ open cover $\{U_\alpha\}$ of X s.t. each $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α.

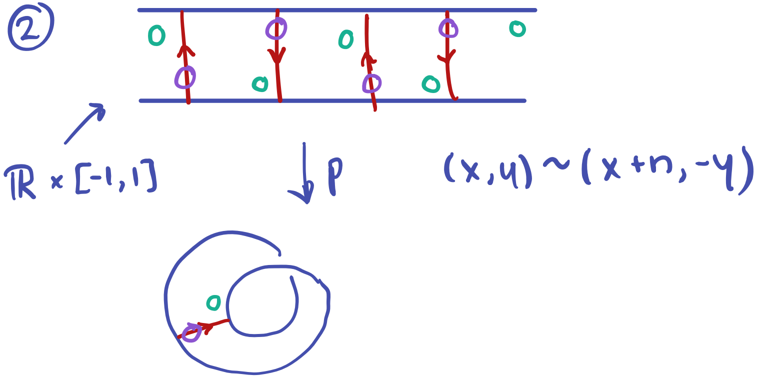


Examples

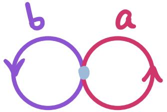
①



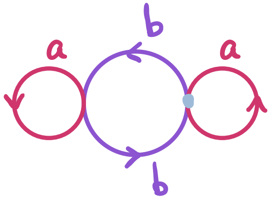
②



Examples

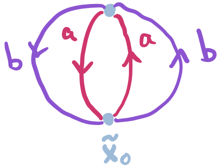
$X =$ 

$\tilde{X}$ | $p_*(\pi_1(\tilde{X})) \le \pi_1(X)$ | $\tilde{X}$ | $p_*(\pi_1(\tilde{X}))$

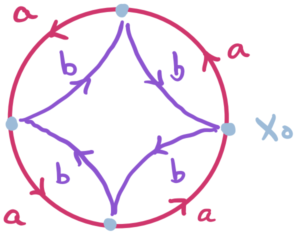


$\langle a, b^2, bab^{-1} \rangle$

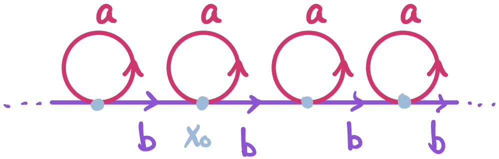
$= \langle a, b^2, bab \rangle$



$\langle ba, b^2, ba^{-1} \rangle$



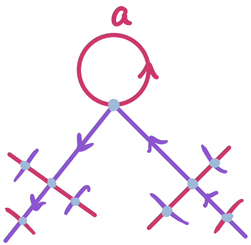
you.



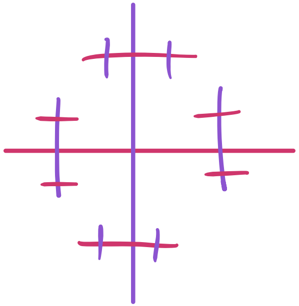
$\langle b^k a b^{-k} \rangle$

Will show p_* inj.

$\Rightarrow F_\infty \le F_2$



$\langle a \rangle$



1

Lifting Properties

$p: \tilde{X} \to X$ cov sp.

A lift of $f: Y \to X$
is an $\tilde{f}: Y \to \tilde{X}$
s.t. $p \circ \tilde{f} = f$
(like lifting paths from S^1 to $\mathbb{R}$).

Prop 1. (Homotopy lifting property)
Given a homotopy $f_t: Y \to X$
and $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0
$\exists!$ $\tilde{f}_t$ lifting f_t.

Special cases ① $Y = pt. \rightsquigarrow f_t$ = path
② Y = interval $\rightsquigarrow f_t$ = homotopy of paths

Pf. Same as S^1.

Cor. $p_*: \pi_1(\tilde{X}) \to \pi_1(X)$ injective.

Pf. If $p_*(\alpha) = 1 \in \pi_1(X) \rightsquigarrow$
homotopy in X to const.
Lift the homotopy $\Rightarrow \alpha = 1$.

Note. $p_*(\pi_1(\tilde{X}))$ is exactly the subgp of $\pi_1(X)$ consisting of loops that lift to loops.

Degree of a cover

$|p^{-1}(x)|$ is locally const (as a fn of $x \in X$) hence const.

This "number" is the degree of p.

Cor. $X, \tilde{X}$ path conn.

$$\deg p = [\pi_1(X) : p_* \pi_1(\tilde{X})]$$

topology　　　group theory.

Pf. Let $H = p_* \pi_1(\tilde{X})$

deg p of these

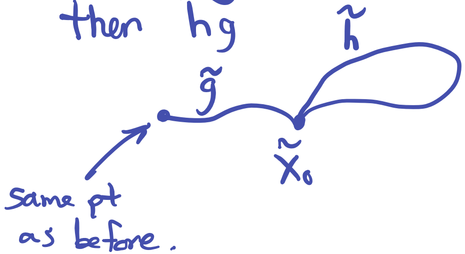
Define $\{\text{right cosets of } H\} \to p^{-1}(x_0)$

$$Hg \longmapsto \tilde{g}(1)$$

Key pt: well def.

If hg is another rep of Hg

then $\widetilde{hg}$



In first graph example, coset reps are:

$$1, \ b$$

So: $\langle a, b^2, bab^{-1} \rangle \leq F_2$

$$\langle a, b^2, bab^{-1} \rangle \cong F_3$$

Index is 2, b is a coset rep.