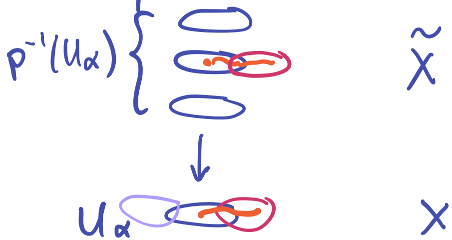
Covering spaces

A covering space of connected X is a connected space $\tilde{X}$ with a map $p: \tilde{X} \to X$ satisfying:

$\exists$ open cover $\{U_\alpha\}$ of X s.t. each $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α.



Lifting Properties

Feb 7

A lift of $f: Y \to X$ is an $\tilde{f}: Y \to \tilde{X}$ s.t. $p \circ \tilde{f} = f$

Prop 1. (Homotopy lifting property)
Given a homotopy $f_t: Y \to X$ and $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0 $\exists!$ $\tilde{f}_t$ lifting f_t.

Special cases: paths, homotopy of paths

Cor. $p_*: \pi_1(\tilde{X}) \to \pi_1(X)$ injective.

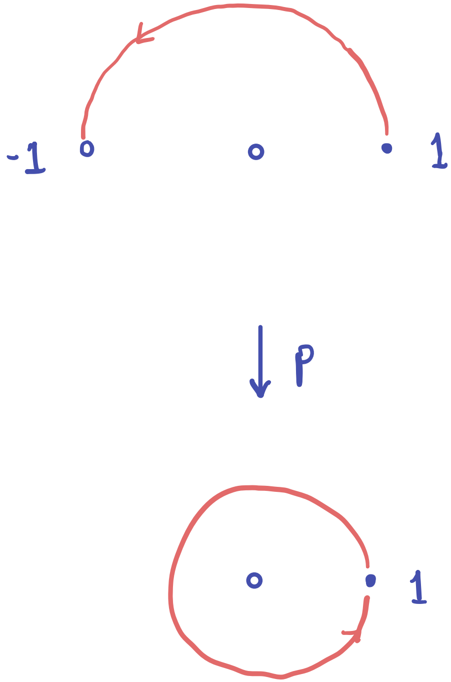
Cor. $X, \tilde{X}$ path conn.

$$\deg p = [\pi_1(X) : p_* \pi_1(\tilde{X})]$$

An important covering space

$$p : \mathbb{C} \setminus 0 \longrightarrow \mathbb{C} \setminus 0$$
$$z \longmapsto z^2$$

Non-uniqueness of square roots corresponds to the fact that this is a nontrivial cover.



$\pi_1(\tilde{X}) = \mathbb{Z}$

$\pi_1(X) = \mathbb{Z}$

$$p_* : \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$
$$p_*(\mathbb{Z}) = 2\mathbb{Z}$$

Prop 2. (Lifting criterion) Y connected, locally path conn. We can lift $f:(Y,y_0)\to(X,x_0)$ to $\tilde f:(Y,y_0)\to(\tilde X,\tilde x_0)$ iff

$$f_*(\pi_1(Y)) \le p_*\pi_1(\tilde X).$$

$$\begin{tikzcd} & \tilde X \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde f"] \arrow[r, "f"'] & X \end{tikzcd}$$

Already know we can lift when $Y = I,\ I\times I$ (special case).

Pf. ($\Rightarrow$) $\tilde f$ exists $\Rightarrow p\circ\tilde f = f$

$\Rightarrow p_*\circ\tilde f_* = f_*$ ✓

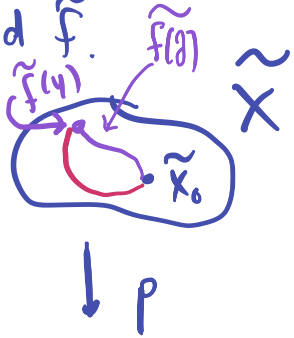
($\Leftarrow$) Suppose $\operatorname{Im} f_* \subseteq \operatorname{Im} p_*$

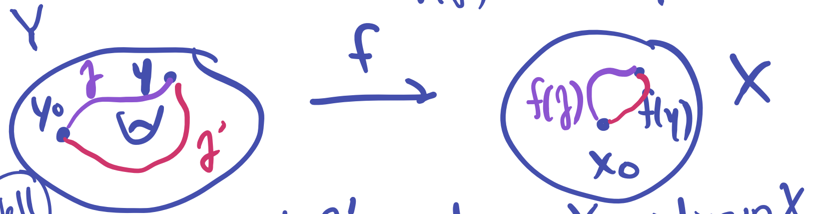
Want to build $\tilde f$.

Let $y\in Y$

Choose path γ, y_0 to y

Define $\tilde f(y) =$ endpt of $\widetilde{f(\gamma)}$





Well def: Another path $\gamma' \rightsquigarrow$ loop in $Y \rightsquigarrow$ loop in X $\rightsquigarrow$ loop in $\tilde X$ (by hypothesis!) $\square$

Prop 3 (Uniqueness of lifts)

Let $f: Y \to X$, Y conn.

If lifts $\tilde{f}_1, \tilde{f}_2$ agree at one pt, then they are equal.

Already know the case $Y = I, I \times I$

Pf. Will show

$$A = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$$

is open & closed in Y.

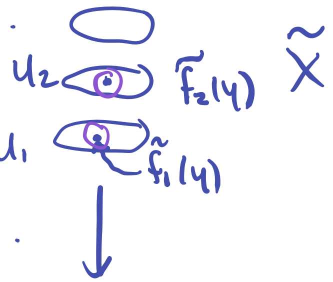
$(\Rightarrow A = Y)$.

not $A \neq \emptyset$ by assumption.

Let $y \in Y$

U = open nbd of $f(y)$ as in defn of cov space.

Let $\tilde{U}_1, \tilde{U}_2$ components of $p^{-1}(U)$ containing $\tilde{f}_1(y), \tilde{f}_2(y)$.



- $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \tilde{U}_1 \neq \tilde{U}_2$

  $\Rightarrow$ open nbd of y not in $A \Rightarrow A$ closed.

- Similarly, $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{U}_1 = \tilde{U}_2$

  $\Rightarrow A$ open. $\square$

Classification of Covering Spaces

$$\left\{\begin{array}{c}\text{based covers of}\\ X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subgps of}\\ \pi_1(X)\end{array}\right\}$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Want a map the other way.

First case: trivial subgp.

__Thm.__ X = CW complex (or any path conn, locally path conn, semilocally simply conn)

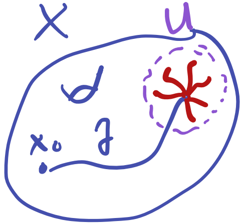
Then X has a universal cover.

__Pf.__ We construct $\tilde{X}$ directly.

Define

$$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$$

$$p : \tilde{X} \to X$$
$$[\gamma] \longmapsto \gamma(1)$$



Topology on $\tilde{X}$:

$$\mathcal{U} = \{U \subseteq X : U \text{ path conn, open, } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

For $U \in \mathcal{U}$, γ with $\gamma(1) \in U$

set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \ \eta(0) = \gamma(1)\}$

Open nbd of $[\gamma]$ in $\tilde{X}$.

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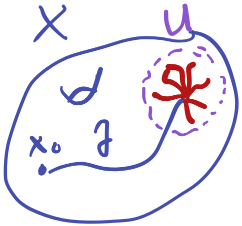

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Open nbd of $[\gamma]$ in $\tilde{X}$.

Check properties of cov sp.

- continuity
- path connectivity.
- If $[\gamma'] \in U_{[\gamma]}$ then

$$U_{[\gamma']} = U_{[\gamma]}$$

Thus, for fixed $U \in \mathcal{U}$ the $U_{[\gamma]}$ partition $p^{-1}(U)$

& $p : U_{[\gamma]} \to U$ is a homeo.

Next time: $\tilde{X}$ simply connected.