Feb 9

RECORD

Classification of Covering Spaces

$$\{\text{based covers of } X\} \longleftrightarrow \left\{\begin{matrix}\text{subgps of} \\ \pi_1(X)\end{matrix}\right\}$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Want a map the other way.

First case: trivial subgp.

Example

$p : \mathbb{R} \to S^1$

Thm. X = CW complex (or any path conn, locally path conn, semilocally simply conn)

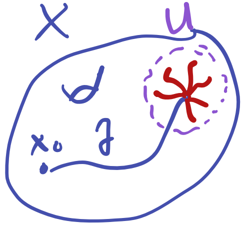
Then X has a universal cover.

Pf. We construct $\tilde{X}$ directly.

Define

$$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$$

$$p : \tilde{X} \to X$$
$$[\gamma] \longmapsto \gamma(1)$$



Topology on $\tilde{X}$:

$$\mathcal{U} = \{U \subseteq X : U \text{ path conn, open}, \ \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

For $U \in \mathcal{U}$, γ with $\gamma(1) \in U$

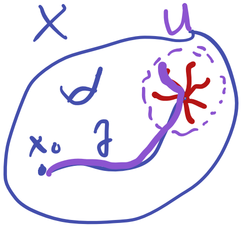
set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \ \eta(0) = \gamma(1)\}$

Open nbd of $[\gamma]$ in $\tilde{X}$.

$\tilde{X} = \{[\gamma] : \gamma \text{ a path in } X \text{ based at } x_0\}$ • Covering space

$p : \tilde{X} \to X$

$[\gamma] \longmapsto \gamma(1)$



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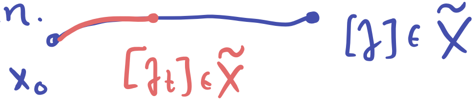
set $U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ path in } U, \eta(0) = \gamma(1)\}$

• p is continuous

$p^{-1}(U)$ is a union of $U_{[\gamma]}$

• $\tilde{X}$ is path conn.



If $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$

$\Longrightarrow$ for fixed U, $\{U_{[\gamma]}\}$ partition $p^{-1}(U)$

and. $p : U_{[\gamma]} \to U$ is a homeo b/c it induces a bijection of open sets

$$V_{[\gamma]} \subseteq U_{[\gamma]} \longleftrightarrow V \subseteq U$$

• Simply connected

p_* inj so enough: $p_* \pi_1(\tilde{X}) = 1$

$\gamma \in \operatorname{Im} p_* \Rightarrow \gamma$ lifts to loop: $\{[\gamma_t]\}$

loop $\Rightarrow [\gamma] = [\gamma_1] \sim [\gamma_0] = \text{const.}$

$\Longrightarrow \gamma = 1$

$\square$

Thm. $\forall\ H \le \pi_1(X)\ \exists$ (based) cov sp $p : \tilde{X}_H \to X$ s.t. $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

Pf. Define $\tilde{X}_H$ as $\tilde{X}/\sim$:

$[\gamma] \sim [\gamma']$ if $\gamma \bar{\gamma}' \in H$

exercise: this is equiv reln.

Check: this is a covering space.

Check: $p_* \pi_1(\tilde{X}_H, \tilde{x}_0) = H$. $\tilde{x}_0 = [\text{const loop}]$

$\gamma \in \operatorname{Im} p_* \iff \{[\gamma_t]\}$ a loop in $\tilde{X}_H$

$\iff [\gamma_0] \sim [\gamma_1] = \gamma$, $\text{id} = \text{const}$

$\iff \gamma \in H$ $\square$

To finish classification, need uniqueness of $\tilde{X}_H$.

Cor. All subgps of free gps are free.

Pf. $F = \pi_1(\Gamma)$ $\Gamma = \text{graph} = \vee S^1$

Any subgp of F is π_1 of a cover of Γ, which is a graph $\square$

Def. Cov sp's $\tilde{X}_1, \tilde{X}_2$ are isomorphic if there is a homeo $f: \tilde{X}_1 \to \tilde{X}_2$ with $p_1 = p_2 f$:

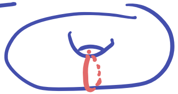
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$$\left(\begin{tikzcd} & \tilde{X}_2 \arrow[d, "p_2"] \\ Y \arrow[ur, dashed, "\tilde{p}_1"] \arrow[r, "p_1"'] & X \end{tikzcd} \right)$$

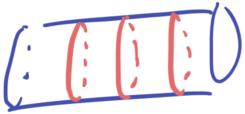
(i.e. f preserves "fibers")

↑ preimage of pt.

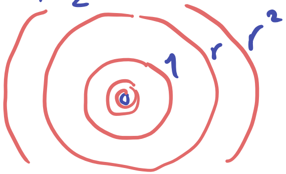
Example

$X =$ 

$\mathrm{Im}(p_1)_* = \mathbb{Z} \times 1$

$\tilde{X}_1 =$  $S^1 \times \mathbb{R}$

$\tilde{X}_2 = \mathbb{C}^*$



Prop. $\tilde{X}_1 \cong \tilde{X}_2 \iff \mathrm{Im}(p_1)_* = \mathrm{Im}(p_2)_*$.

Pf. ($\Rightarrow$) easy. (use: f_* is $\cong$).

($\Leftarrow$) Lifting criterion $\rightsquigarrow$

lift p_1 to $\tilde{p}_1$: $p_2 \tilde{p}_1 = p_1$

by symmetry: $p_1 \tilde{p}_2 = p_2$

Note: $\tilde{p}_1 \tilde{p}_2$ is a lift of p_2

$p_2 \tilde{p}_1 \tilde{p}_2 = p_1 \tilde{p}_2 = p_2$

Uniqueness of lifting + $\tilde{p}_1 \tilde{p}_2(\tilde{x}_2) = \tilde{x}_2$

$\Rightarrow \tilde{p}_1 \tilde{p}_2 = id$

$\Rightarrow$ both homeos. $\square$

The Fundamental Thm

Fix $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$

$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$N(H)$ = normalizer of H

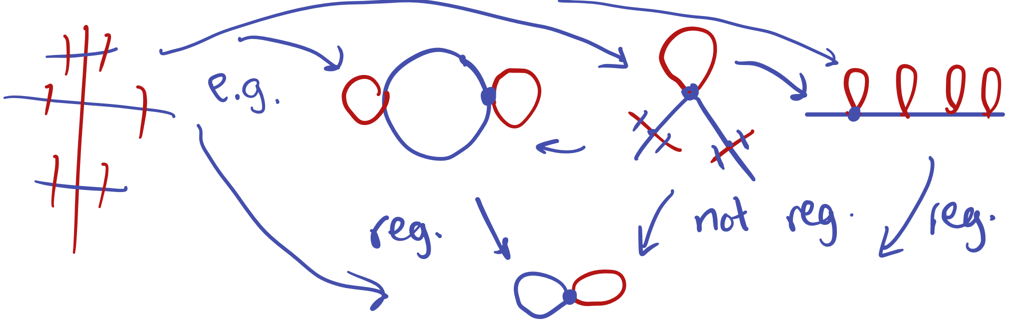
$= \{g : gHg^{-1} = H\}$

$G(\tilde{X})$ = gp of deck transf.

= gp of isomorphisms of $\tilde{X}$ ↻

Say p is regular if $G(\tilde{X})$ acts transit. on $p^{-1}(x_0)$.

e.g.



Prop. $\tilde{X}$ reg $\iff$ H normal.

Thm. $G(\tilde{X}) \cong N(H)/H$

i.e.

$$1 \to H \to N(H) \to G(\tilde{X}) \to 1$$