Last time we proved:

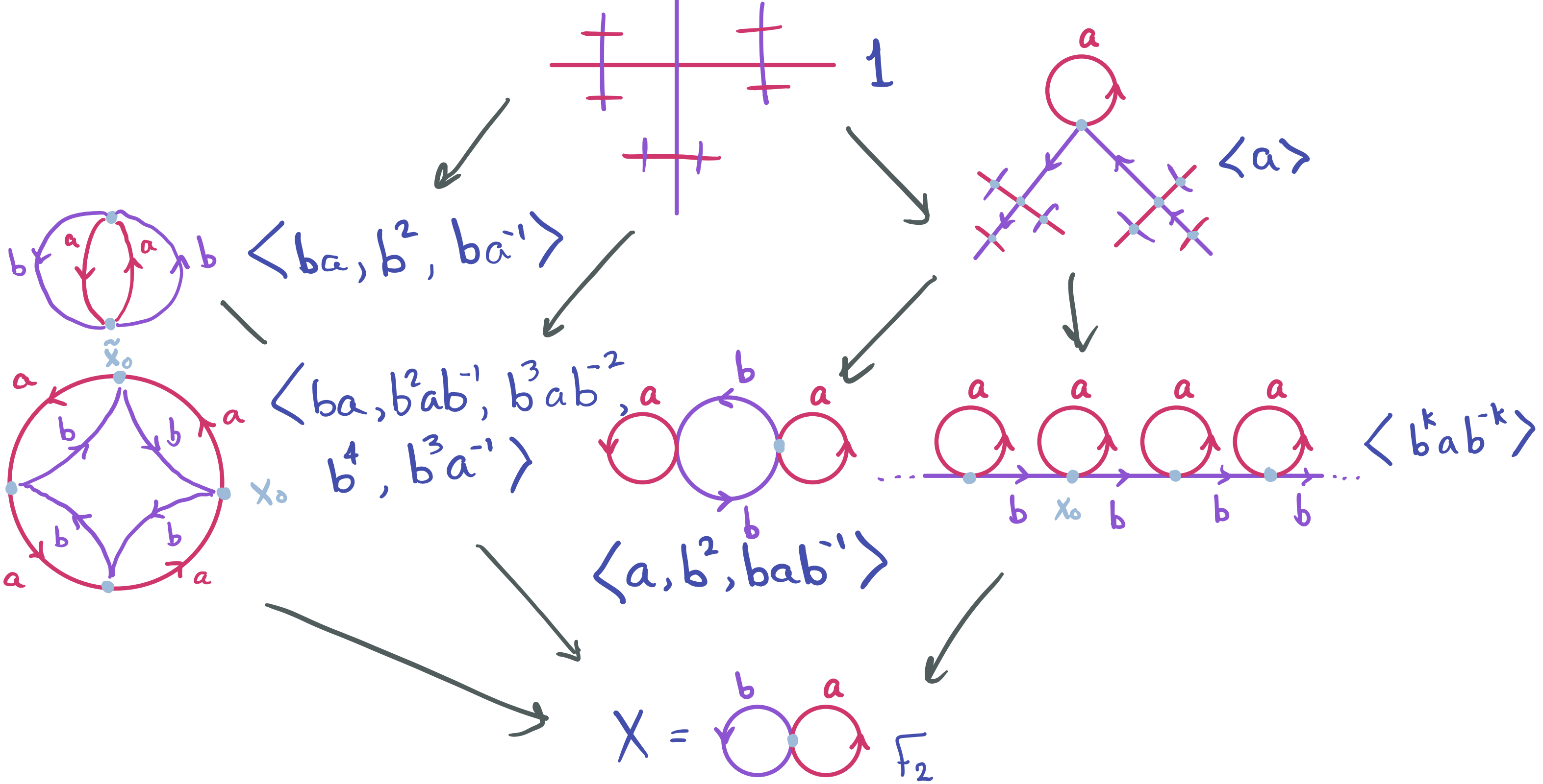
$$\left\{ \begin{matrix} \text{based cov sp's} \\ \text{of } X \end{matrix} \right\}_{/\sim} \longleftrightarrow \left\{ \begin{matrix} \text{subgps of} \\ \pi_1(X) \end{matrix} \right\}$$

It follows that the poset of subgps corresponds to the poset of covers:

$$\begin{tikzcd}
 & \pi_1(X) & \\
A \arrow[ur, hook] & & B \arrow[ul, hook'] \\
C \arrow[u, hook] \arrow[urr, hook] & & D \arrow[u, hook'] \\
 & 1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
 & \tilde{X} \arrow[dl] \arrow[dr] & \\
\tilde{X}_C \arrow[d] \arrow[drr] & & \tilde{X}_D \arrow[d] \\
\tilde{X}_A \arrow[dr] & & \tilde{X}_B \arrow[dl] \\
 & X &
\end{tikzcd}$$

"contravariant functor"

Examples



1
$\langle a \rangle$
$\langle ba, b^2, ba^{-1} \rangle$
$\langle ba, b^2ab^{-1}, b^3ab^{-2}, b^4, b^3a^{-1} \rangle$
$\langle a, b^2, bab^{-1} \rangle$
$\langle b^k a b^{-k} \rangle$
$\tilde{x}_0$
x_0
$X =$
F_2

Isomorphisms

Cov sp's $\tilde{X}_1, \tilde{X}_2$ are isomorphic

if there is:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

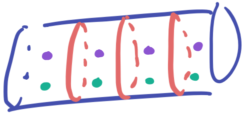
preimage of pt.

(i.e. f preserves "fibers")

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y \arrow[ur, dotted] \arrow[r] & X \end{tikzcd}$$

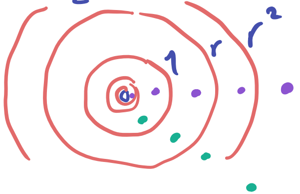
Example

$X =$ 

$\tilde{X}_1 =$  $S^1 \times \mathbb{R}$

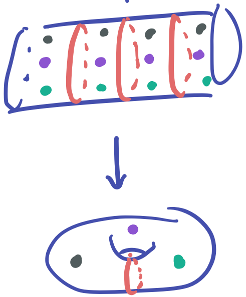
$\mathrm{Im}(p_1)_*$ $=$ $\mathbb{Z} \times 1$

$\tilde{X}_2 = \mathbb{C}^*$



Deck Transformations

A deck transformation of a cover is an automorphism (self-isomorphism). $G(\tilde{X}) = \{\text{deck t's}\}$

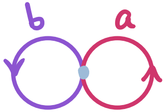


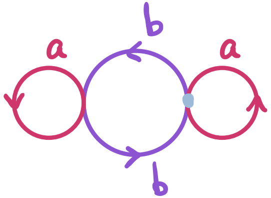
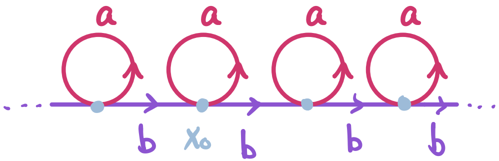
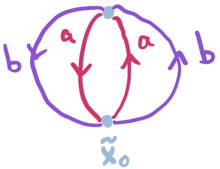
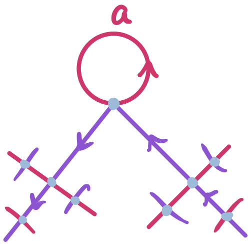
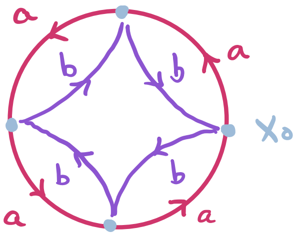
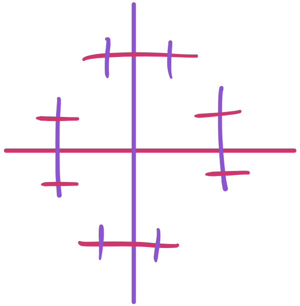
What are all the deck transf's?

Translation by $\mathbb{Z}$

Uniqueness of lifts $\Longrightarrow$ Deck T's determined by one pt.

Examples

$X =$ 

$\tilde{X}$	$G(\tilde{X})$		$\tilde{X}$	$G(\tilde{X})$
	$\mathbb{Z}/2$	rotate by π.		$\mathbb{Z}$
	$\mathbb{Z}/2$	rotate by π		1
	$\mathbb{Z}/4$	rotate by $\pi/2$		F_2

The Fundamental Thm

Fix $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$

$$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong \pi_1(\tilde{X})$$

$N(H)$ = normalizer of H

$$= \{\gamma : \gamma H \gamma^{-1} = H\} \subseteq \pi_1(X).$$

There is a map:

$$F : N(H) \longrightarrow G(\tilde{X})$$

$\gamma \longmapsto$ deck transf taking $\tilde{x}_0$ to $\tilde{\gamma}(1)$.

Thm. F is well-def & surjective, and $\ker F = H$. In other words:

$$1 \longrightarrow H \hookrightarrow N(H) \xrightarrow{F} G(\tilde{X}) \longrightarrow 1$$

is exact. In particular:

$$N(H)/H \cong G(\tilde{X}).$$

Cor. If $H \trianglelefteq \pi_1(X)$ then

$$\pi_1(X)/\pi_1(\tilde{X}) \cong G(\tilde{X}).$$

<u>Key pt</u>: F is well def.

Regard $\tilde{x}_0$ as $[\text{const}]$.

Then $p^{-1}(x_0) = \{[\gamma] : \gamma \text{ a loop}\}/\sim$

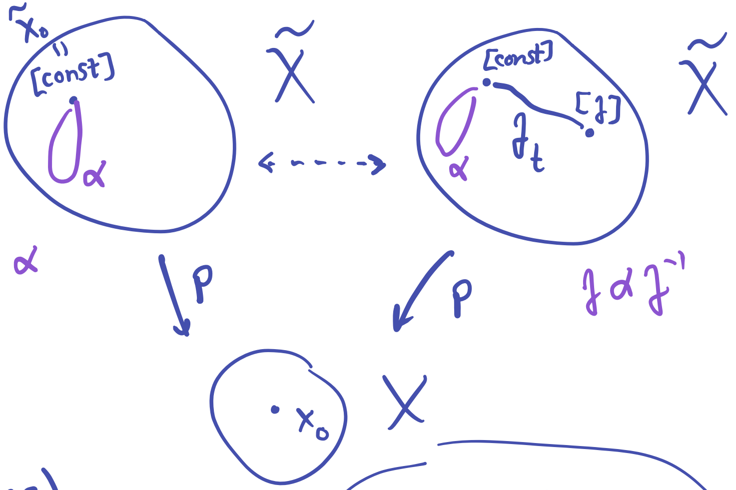
By lifting criterion:

$\exists$ deck tr τ taking $[\text{const}]$ to $[\gamma]$

$\iff p_*\pi_1(\tilde{X}, [\text{const}]) \overset{=}{\cancel{\subseteq}} p_*\pi_1(\tilde{X}, [\gamma])$.

$\iff \gamma\, p_*\pi_1(\tilde{X}, [\text{const}])\, \gamma^{-1} = p_*\pi_1(\tilde{X}, [\text{const}])$

$\iff \gamma \in N(H)$.



<u>Remark</u>: If $[\gamma] = [\text{const}]$ then deck transf is trivial.

This argument also shows surjectivity: Given $\tau \in G(\tilde{X})$. Let $\tilde{\gamma}$ be path $\tilde{x}_0$ to $\tau\cdot x_0$. Let $\gamma = p\circ\tilde{\gamma}$.

Examples

$$F_2 / \langle a, bab^{-1}, b^2 \rangle \cong \mathbb{Z}/2$$

$$F_2 / \langle b^k a b^{-k} \rangle \cong \mathbb{Z}$$

Regular cov sp: $G(\tilde{X})$ acts trans. on $p^{-1}(x_0)$.

Prop. $\tilde{X}$ regular

$\iff H$ normal

$(\iff N(H) = \pi_1(X))$

Covering spaces via actions

An action of a gp G on a space Y is:

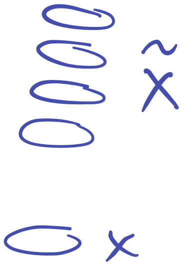
$$G \to \mathrm{Homeo}(Y)$$

This is a cov sp action if

$\forall\, y \in Y \;\exists$ nbd U s.t.

$\{g(U)\}$ all distinct, disjoint.

Fact. The action of $G(\tilde{X})$ on $\tilde{X}$ is a cov sp action



Prop. Y = conn CW complex

$G \curvearrowright Y$ cov sp action

Then ① $p : Y \to Y/G$ reg cov sp.

② $G \cong G(Y)$.

Example