Feb 14

Covering spaces via actions

An action of a gp G on a space Y is:

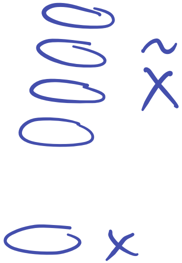
$$G \to \mathrm{Homeo}(Y)$$

This is a **cov sp action** if

$\forall y \in Y\ \exists$ nbd U s.t.

$\{g(U)\}$ all distinct, disjoint.

Fact. The action of $G(\tilde{X})$ on $\tilde{X}$ is a cov sp action



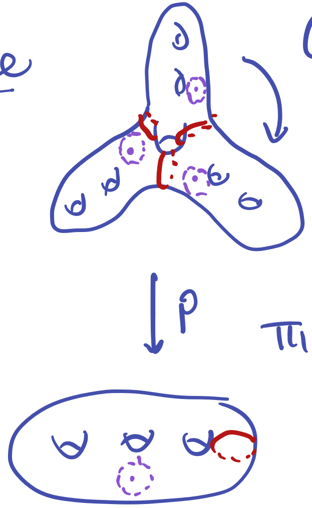
Prop. Y = conn CW complex

$G \curvearrowright Y$ cov sp action

Then ① $p : Y \to Y/G$ reg cov sp.

② $G \cong G(Y)$.

Example



$\pi_1(S_7) \leq \pi_1(S_3)$

Prop. Y = conn CW complex

$G \curvearrowright Y$ cov sp action

Then ① $p: Y \to Y/G$ reg cov sp.

② $G \cong G(Y)$.

In particular:

$$G \cong \pi_1(Y/G)/p_* \pi_1(Y)$$

Further: Y simply connected

$$G \cong \pi_1(Y/G).$$

Examples of cov sp actions

① $\mathbb{Z}/2 \curvearrowright S^2$

$\implies \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$

Actually works for $\mathbb{Z}/2 \curvearrowright S^n$

$\implies \pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$

② $\mathbb{Z} \curvearrowright \mathbb{R}$

$\implies \pi_1(S^1) = \mathbb{Z}$

Actually: $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$

$\rightsquigarrow \pi_1(T^n) \cong \mathbb{Z}^n$

$K(G,1)$ Spaces

Goal: groups $\longleftrightarrow$ spaces/$\sim$

homoms $\longleftrightarrow$ maps

Def. A $K(G,1)$ is a space X

with ① $\pi_1(X) \cong G$

② $\tilde{X}$ contractible.

(contractible universal cover)

Examples S^1, T^n, $S^1 \vee S^1$

$\quad\ \ $ " $\quad$ " $\quad$ "

$K(\mathbb{Z},1)$ $K(\mathbb{Z}^n,1)$ $K(F_2,1)$

Note: $\mathbb{RP}^2$ is not a $K(\mathbb{Z}/2,1)$.

What about $\mathbb{Z}/m\mathbb{Z}$?

$$\mathbb{Z}/m\mathbb{Z} \curvearrowright S^\infty \subseteq \mathbb{C}^\infty$$

$$(z_i) \longmapsto e^{2\pi i/m}(z_i)$$

This is a cov sp. action.

$\Rightarrow$ quotient is $K(\mathbb{Z}/m,1)$,

since $S^\infty \simeq *$.

Later: Any $K(\mathbb{Z}/m,1)$ is

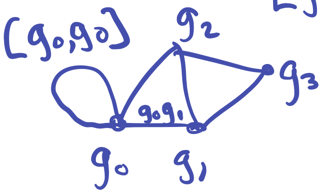
∞-dim'l!

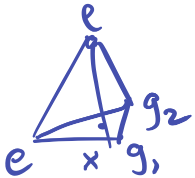
Construction of $K(G,1)$ spaces

Prop. Every group G has a $K(G,1)$.

Pf. Define Δ-complex EG with

n-simplices $\longleftrightarrow$ ordered $(n+1)$-tuples $[g_0, \ldots, g_n]$ $g_i \in G$.





Claim. EG is contractible.

Pf. Slide each $x \in [g_0, \ldots, g_n]$ along line segment in $[e, g_0, \ldots, g_n]$ to e.

Not a def ret b/c. $[e]$ moves along $[e,e]$

Now: $G \circlearrowright EG$ by left mult.

Claim: This is a cov sp. action

Pf: you!

So: $BG = EG/G$ is a $K(G,1)$. $\square$

This gives an ∞-dim $K(G,1)$.

Often: goal is to find the <u>right</u> $K(G,1)$.

Homomorphisms as maps

Prop. X = connected CW complex
$Y = K(G,1)$ maybe a $K(H,1)$.
Every homomorphism $\pi_1(X,x_0) \to G$ is induced by a map $X \to Y$. The map is unique up to homotopy.

Examples ① $\pi_1(M_g) \to \mathbb{Z}^{2g}$ $g \geq 1$ abelianization.
$\rightsquigarrow M_g \to T^{2g}$ (Jacobian map).

② Any $\mathbb{RP}^2 \to T^2$ homotopic to const.
$\mathbb{Z}/2 \to \mathbb{Z}^2$ (must be trivial)

Cor. $K(G,1)$ spaces are unique up to homotopy equiv.

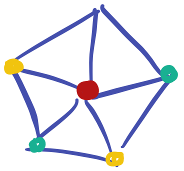
Pf. If X, Y are $K(G,1)$'s, the identity $G \to G$ gives
$X \to Y$
$Y \to X$. $\square$

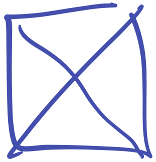
Idea of Pf of Prop Given $f: \pi_1(X) \to G$
Assume with loss of gen. X has one vertex.

0-cells. Send x_0 to y_0
1-cells. Determined by f.
2-cells Det. by f.
uniqueness: since $\tilde{Y} \simeq *$. $\square$

Application of Cov spaces

Thm. A convex polytope made of triangles is 3-colorable $\Longleftrightarrow$ even # of triangles at each vertex.

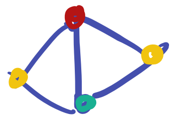




Pf (Kontsevich).

For each triangle, color it all 6 ways.

Glue together when sides match and triangles match in original polytope



This is a cov. space. (actually, every connected component is).

Each connected component is trivial cover. $\square$