Feb 16

Homology

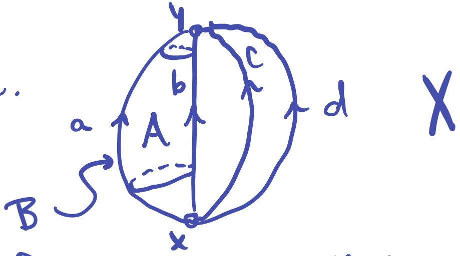
π_1 is useful, but hard to compute.

π_i is harder to compute:

$\pi_m(S^n)$ is a major open problem.

Homology is a computable version...

Example.



C_0 = free abel gp on x, y.

C_1 = free abel gp on a, b, c, d

C_2 = free abel gp on A, B.

$c - d = -d + c$

is the <u>unbased</u> clockwise loop around c & d.

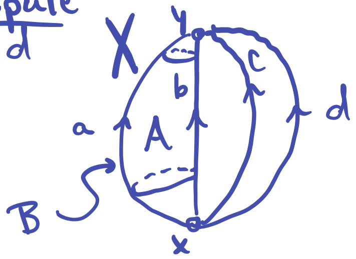
An elt of $H_1(X)$ is a <u>1-cycle</u>: an elt of C_1 with no boundary.

Two are equiv. if they differ by boundary of elt of C_2.

So $H_1(X) = $ 1-cycles / 1-boundaries.

e.g. $a - b = \partial A \Rightarrow a - b \sim 0$.

Compute X



X

$\partial_0 : C_0 \longrightarrow 0$

"boundary map"

$\partial_1 : C_1 \longrightarrow C_0$

$a, b, c, d \longmapsto y - x$

$\partial_2 : C_2 \longrightarrow C_1$

$A, B \longmapsto a - b$

So $H_1(X) =$ 1-cycles / 1-boundaries

$= \ker \partial_1 / \mathrm{Im}\, \partial_2$

exercise

$= \langle a-b, b-c, c-d \rangle / \langle a-b \rangle$

$\cong \mathbb{Z}^2$

$H_2(X) = \ker \partial_2 / \mathrm{Im}\, \partial_3$

$= \mathbb{Z}.$

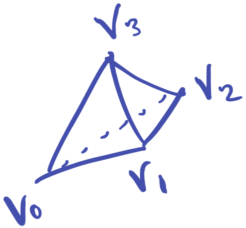
$H_0(X) = \ker \partial_0 / \mathrm{Im}\, \partial_1$

$= \langle x, y \rangle / \langle y - x \rangle$

$= \langle x, y-x \rangle / \langle y - x \rangle \cong \mathbb{Z}$

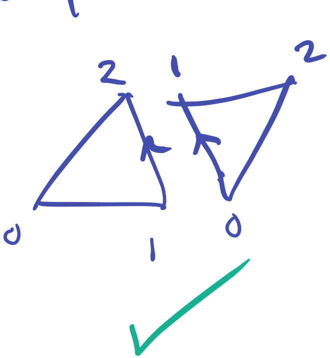
Δ-complex

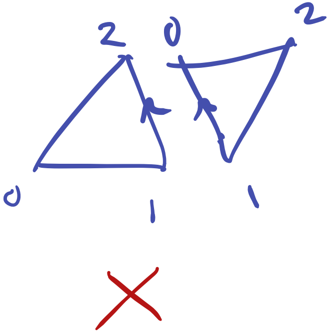
Ordered Simplex :



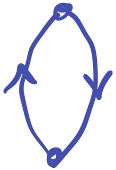
order the vertices on each simplex

A Δ-complex is obtained from a collection of ordered simplices by gluing faces in order preserving way:



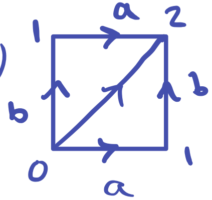


Not a Δ-complex :

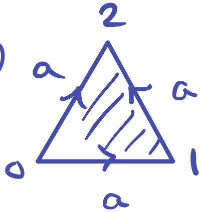


Examples

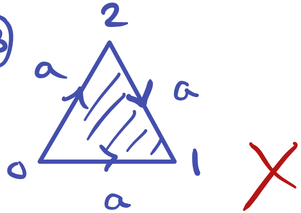
①



②



③

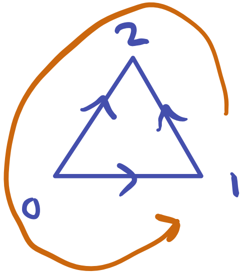


Can subdivide and make it a Δ-complex (exercise).

Boundaries

$$\partial([v_0,\dots,v_n]) = \sum (-1)^i [v_0,\dots,\hat{v}_i,\dots,v_n]$$

e.g. $\partial([v_0,v_1,v_2]) = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$



exercise: think about $\partial($  $)$.

Lemma. $\partial_{n-1} \circ \partial_n = 0$.

Pf. Check on each simplex

$$\partial_{n-1}\partial_n([v_0,\dots,v_n]) =$$

$$\partial_{n-1}\Big(\sum (-1)^i [v_0,\dots,\hat{v}_i,\dots,v_n]\Big)$$

$$= \sum_{j<i} (-1)^{i+j} [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$$

$$+ \sum_{i<j} (-1)^{i+j-1} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$$

$$= 0. \qquad \square$$

In other words: $\operatorname{Im}\partial_n \subseteq \ker \partial_{n-1}$

We now have:

$$\cdots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

where $\Delta_i(X)$ = free abel gp on i-simplices

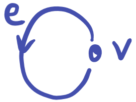
and. $\operatorname{Im} \partial_n \subseteq \ker \partial_{n-1}$.

(prev lemma).

So, It makes sense to define

$$H_k(X) = \ker \partial_{k-1} / \operatorname{Im} \partial_k$$

$$= k\text{-cycles} / k\text{-boundaries}.$$

Examples ① $X = S^1$ 

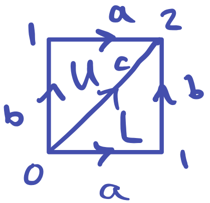
$$\Delta_0(X) = \langle v \rangle \cong \mathbb{Z}$$

$$\Delta_1(X) = \langle e \rangle \cong \mathbb{Z}$$

$$\partial_1 = 0 \qquad \partial_1(e) = v - v = 0.$$

$$\Longrightarrow H_k(X) = \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & \text{otherwise}. \end{cases}$$

② $X = T^2$



$\partial_1 = 0. \quad \partial_0 = \partial_3 = 0.$

$\partial_2(U) = \partial_2(L) = a+b-c$

$H_0(X) = \mathbb{Z}/0 \cong \mathbb{Z}$

$H_1(X) = \langle a,b,c\rangle / \langle a+b-c\rangle \cong \mathbb{Z}^2$

$H_2(X) = \langle U-L\rangle / 0 \cong \mathbb{Z}$

$U-L$ is the torus.