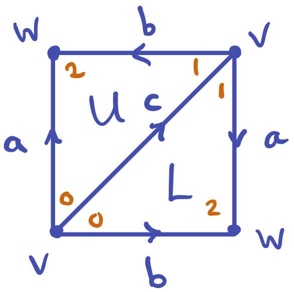
③ $X = \mathbb{RP}^2$



Feb 18

$$\begin{array}{ccccccc}
\Delta_2 & \xrightarrow{\partial_2} & \Delta_1 & \xrightarrow{\partial_1} & \Delta_0 & \xrightarrow{\partial_0} & 0 \\
\cong & & \cong & & \cong & & \\
\mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z}^2 & & \\
\parallel & & \parallel & & \parallel & & \\
\langle U, L\rangle & & \langle a,b,c\rangle & & \langle v,w\rangle & &
\end{array}$$

$$\partial_2 : \quad \begin{aligned} U &\longmapsto -a+b+c \\ L &\longmapsto a-b+c \end{aligned}$$

$$\partial_1 : \quad \begin{aligned} a &\longmapsto w-v \\ b &\longmapsto w-v \\ c &\longmapsto 0 \end{aligned}$$

$$\partial_0 \quad v, w \longmapsto 0$$

$$H_0(X) = {\ker \partial_0}/{\operatorname{im} \partial_1} = \langle v, w\rangle / \langle w - v\rangle = \mathbb{Z}$$

$$H_1(X) = {\ker \partial_1}/{\operatorname{im}\partial_2}$$

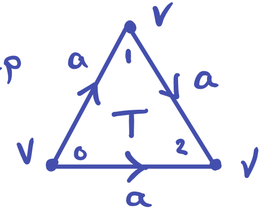
$$= \langle a-b, c\rangle / \langle -a+b+c, a-b+c\rangle$$

$$\langle a-b, c\rangle = \langle c, a-\cancel{b+c}\rangle \qquad \langle -a+b+c, a-b+c\rangle = \langle 2c, a-\cancel{b+c}\rangle$$

$$= \langle c\rangle / \langle 2c\rangle \cong \mathbb{Z}/2$$

$$H_2(X) = {\ker \partial_2}/{\operatorname{im}\partial_3} = 0/0 = 0.$$

④ X = Dunce cap



(X is contractible but not collapsible)

$$\begin{array}{ccccccc} \Delta_2 & \longrightarrow & \Delta_1 & \longrightarrow & \Delta_0 & \longrightarrow & 0. \\ \parallel & & \parallel & & \parallel & & \\ \langle T \rangle & & \langle a \rangle & & \langle v \rangle & & \end{array}$$

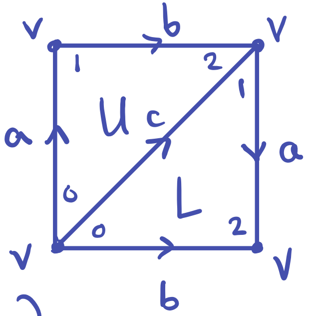
$$T \longmapsto a \longmapsto 0$$
$$v \longmapsto 0$$

$H_0(X) = \langle v \rangle / 0 = \mathbb{Z}$

$H_1(X) = \langle a \rangle / \langle a \rangle = 0$

$H_2(X) = 0/0 = 0.$

⑤ X = Klein bottle.



$5a + 7b$
$= 5(a+b) + 2b = 5c + 2b$

$H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2$

$= \langle a, b, c \rangle / \langle a+b-c, a-b+c \rangle$

What abelian gp is this?

Answer: Smith normal form.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{\text{col op}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow[\text{op}]{\text{row}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$ each diag divides next

$$\rightsquigarrow \mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/0\mathbb{Z} = 0 \times \mathbb{Z}/2 \times \mathbb{Z}$$

Will show: $H_1(X) = \pi_1(X)^{ab}$

Exercise: $H_1(M_g) = \mathbb{Z}^{2g}$

Exact Sequences

A seq. of homoms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is exact if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$

is a chain complex if $\ker \alpha_n \supseteq \operatorname{im} \alpha_{n+1}$

i.e. $\alpha_n \circ \alpha_{n+1} = 0$.

Facts

(i) $0 \to A \xrightarrow{\alpha} B$ exact $\iff$ α inj.

(ii) $A \xrightarrow{\alpha} B \to 0$ exact $\iff$ α surj

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ exact $\iff$ α is $\cong$

(iv) $0 \to A \xrightarrow{\alpha} B \to C \to 0$ exact $\iff C \cong B/A$

where A is identified with subgp of B by the map α

$$\left(0 \to \mathbb{Z} \xrightarrow{\times 6} \mathbb{Z} \to \mathbb{Z}/6 \to 0 \implies \text{"}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}/6\text{"} \right).$$

Four Theorems

① Long exact seq. for collapsing a subcomplex.

② Long ex. seq. for a pair

③ Excision

④ Mayer-Vietoris.

① Collapsing a Subcomplex

Thm. (X,A) is a CW pair

There is an exact seq.

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0$$

where $i: A \hookrightarrow X$
$q: X \to X/A$.

Cor. $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{otherwise}. \end{cases}$

Pf. Take $X = D^n$, $A = S^{n-1}$ $\rightsquigarrow$ $X/A = S^n$

Induction on n.

$\tilde{H}_0(S^0) = \mathbb{Z}$.

Let $n>0$. By Thm:

$$\cdots \to \underbrace{\cancel{\tilde{H}_i(D^n)}}_{0} \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1}) \to \underbrace{\cancel{\tilde{H}_{i-1}(D^n)}}_{0}$$

$$\Longrightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$