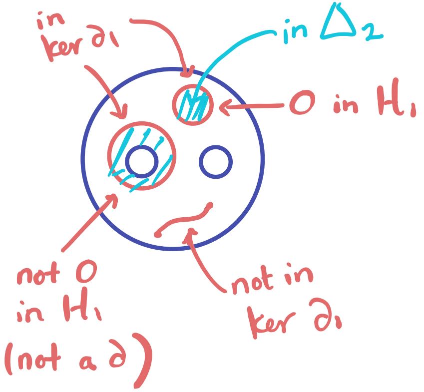
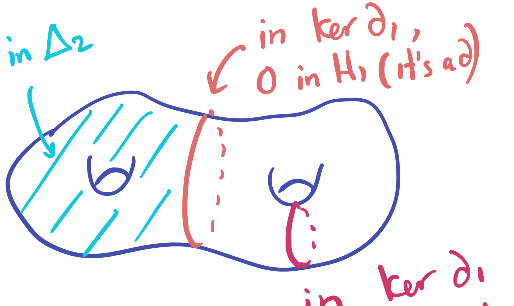
# Simplicial Homology

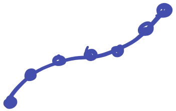
$$\cdots \longrightarrow \Delta_{n+1} \xrightarrow{\partial_{n+1}} \Delta_n \xrightarrow{\partial_n} \Delta_{n-1} \longrightarrow \cdots$$

Fact. $\mathrm{Im}\,\partial_{n+1} \subseteq \ker \partial_n$ (or: $\partial_n \circ \partial_{n+1} = 0$)

$$\rightsquigarrow H_n(X) = \ker \partial_n / \mathrm{Im}\,\partial_{n+1}$$



# Four Theorems

① Long exact seq. for collapsing a subcomplex.

② Long ex. seq. for a pair

③ Excision

④ Mayer-Vietoris.

# Singular Homology

Simplicial hom. computable but

① Not obvious that homeomorphic spaces have same homology.

② Hard to prove general thms.

So: A singular $n$-simplex in $X$ is a map $\sigma : \Delta^n \to X$

Let $C_n(X)$ = free abel. gp on these.

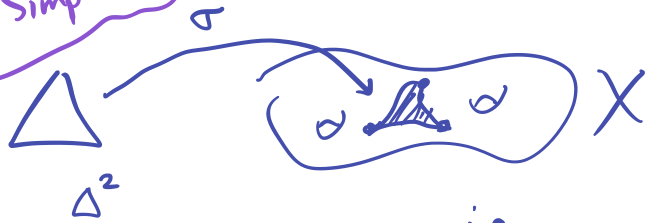
= gp of $n$-chains

$= \left\{ \sum n_i \sigma_i : n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to \mathbb{Z} \right\}$

Boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\sigma \longmapsto \sum (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

We'll show: Singular = Simplicial




Still have: $\partial_{n+1} \circ \partial_n = 0$ i.e. $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$

$\rightsquigarrow H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$.

"singular homology"

Hard to compute. Not obvious that

① $H_k(X) = 0 \quad k > \dim X$

② $H_k(X)$ is ever countable.

**Prop.** $X$ = space with path components $X_\alpha$

$$\Rightarrow H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

**Prop.** $X$ = nonempty, path conn $\Rightarrow H_0(X) = \mathbb{Z}$

$X$ has $n$ path comp $\Rightarrow H_0(X) = \mathbb{Z}^n$

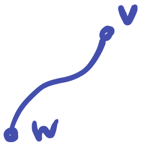
**Pf.** By first Prop. suffices to prove $1^{st}$ statement.

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$C_0(X)$: free abel. gp on pts in $X$

$$H_0(X) = \ker \partial_0 / \operatorname{Im} \partial_1$$

$$= C_0(X)/\operatorname{Im}\partial_1$$




$\operatorname{Im}\partial_1 = \{v - w : v, w \text{ connected by path}\}$ $\square$

**Prop.** $X = pt \Rightarrow$

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

**Pf.** $C_n(X) \cong \mathbb{Z} \quad \forall n.$

$$\partial(\sigma_n) = \sum (-1)^i \sigma_{n-1}$$

$$= \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even.} \end{cases}$$

$$C_4 \to C_3 \to C_2 \to C_1 \to C_0$$

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \boxed{\mathbb{Z}} \to 0$$

$\ker = \mathbb{Z}/\operatorname{Im}\mathbb{Z} = 0$

$\ker = 0$

$\ker/\operatorname{Im} = \mathbb{Z}$ $\square$

# Reduced Homology

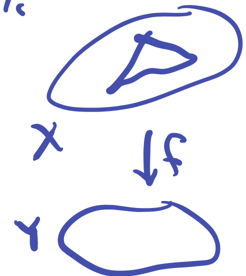
$$\tilde{H}_n(X) = \begin{cases} H_n(X)/\mathbb{Z} & n=0 \\ H_n(X) & n>0 \end{cases}$$

Can achieve this by replacing the last $0$ map above by "evaluation map" $\varepsilon$

$$C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$\varepsilon\left(\sum n_i v_i\right) = \sum n_i$$

$\rightsquigarrow$ same $\tilde{H}$ as above.




# Homotopy Invariance

Goal: $f: X \to Y \rightsquigarrow f_*: H_n(X) \to H_n(Y)$

& $f$ a hom. eq. $\Rightarrow f_*$ an isom.

First: $f \rightsquigarrow f_\#: C_n(X) \to C_n(Y)$

$$\sigma \longmapsto f \circ \sigma$$

Have: $f_\# \partial = \partial f_\#$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$f_\#$ called a chain map. It takes cycles to cycles, $\partial$ to $\partial$

$\rightsquigarrow$ induced map

$$f_* : H_n(X) \to H_n(Y).$$

__Facts.__ $(fg)_* = f_* g_*$

$\mathrm{id}_* = \mathrm{id}$

__Thm.__ $f, g : X \to Y$ homotopic

$$\Rightarrow f_* = g_*$$

__Cor.__ $f : X \to Y$ hom eq

$$\Rightarrow f_* \text{ is } \cong$$

__Example.__ $X \simeq * \Rightarrow H_*(X) = 0$

## ① Collapsing a Subcomplex

**Thm.** $(X,A)$ is a CW pair

There is an exact seq.

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A)$$

$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$$

$$\to \tilde{H}_0(X/A) \to 0$$

where $i: A \hookrightarrow X$

$q: X \to X/A$.

To prove the Thm will do something more general.

**Cor.** $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{otherwise.} \end{cases}$

**Pf.** Take $X = D^n$, $A = S^{n-1}$ $\rightsquigarrow$ $X/A = S^n$

Induction on $n$.

$\tilde{H}_0(S^0) = \mathbb{Z}$.

Let $n>0$. By Thm:

$$\cdots \to \underbrace{\cancel{\tilde{H}_i(D^n)}}_{0} \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1}) \to \underbrace{\cancel{\tilde{H}_{i-1}(D^n)}}_{0}$$

$$\Longrightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$