Homotopy Invariance

Feb 23

Goal: $f : X \to Y \rightsquigarrow f_* : H_n(X) \to H_n(Y)$

& f a hom. eq. $\Rightarrow f_*$ an isom.

First: $f \rightsquigarrow f_\# : C_n(X) \to C_n(Y)$

$$\sigma \longmapsto f \circ \sigma$$

Have: $f_\# \partial = \partial f_\#$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \overset{c}{C_{n+1}(X)} \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & \overset{0}{C_n(X)} \arrow[r, "\partial"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & \overset{f_\#(c)}{C_{n+1}(Y)} \arrow[r, "\partial"] & \overset{0}{C_n(Y)} \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

$f_\#$ called a chain map. It takes cycles to cycles, ∂ to ∂

$\rightsquigarrow$ induced map

$$f_* : H_n(X) \to H_n(Y).$$

Facts. $(fg)_* = f_* g_*$

$\mathrm{id}_* = \mathrm{id}$

Thm. $f, g : X \to Y$ homotopic

$\Rightarrow f_* = g_*$

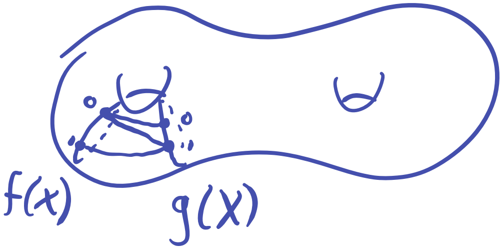
Cor. $f : X \to Y$ hom eq

$\Rightarrow f_*$ is $\cong$

Thm. $f, g : X \to Y$ homotopic

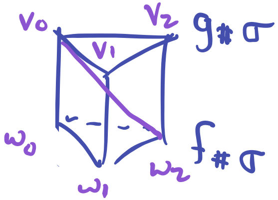
$$\Rightarrow f_* = g_*$$

Idea. $X = S^1 \quad Y = M_2$



$f_* = g_*$ means $f_*(S^1) - g_*(S^1)$ is a boundary. In the example, it is the boundary of a cylinder.

Pf. For σ any singular n-simplex in X. The homotopy gives a prism in Y



Label the vertices on top by $v_0, \dots, v_n$

bot by $w_0, \dots, w_n$

$\Delta^n \times I$ decomposes as sum of

$$[v_0, \dots, v_i, w_i, \dots, w_n]$$

Call this sum $P(\sigma)$

Check $\partial(\Delta^n \times I) = \text{top} - \text{bottom}$.

$\square$

Algebraically: We just defined a map $P: C_n(X) \to C_{n+1}(Y)$

$$\sigma \longrightarrow P(\sigma)$$

with $\partial P = \underbrace{g_\#}_{\text{top}} - \underbrace{f_\#}_{\text{bot}} - \underbrace{P\partial}_{\text{sides}}$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \overset{c}{C_{n+1}(X)} \arrow[r, "\partial"] \arrow[d, "f_\#"] & \overset{0}{C_n(X)} \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & \overset{f_\#(c)}{C_{n+1}(Y)} \arrow[r, "\partial"] & \overset{0}{C_n(Y)} \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Chain homotopy.

The thm follows:

If $\alpha \in C_n(X)$ is a cycle then

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + \cancel{P\partial(\alpha)}$$

$\Longrightarrow (g_\# - f_\#)(\alpha)$ is a ∂

$\Longrightarrow g_*(\alpha) = f_*(\alpha)$.

$\square$

<u>Four goals</u>

1. Contracting subcomplex

1'. Long ex seq for a pair

2. Excision

3. Mayer-Vietoris.

① Collapsing a Subcomplex

Thm. (X,A) is a CW pair

There is an exact seq.

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A)$$

$$\xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{q_*} \tilde{H}_{n-1}(X/A)$$

$$\cdots \xrightarrow{q_*} \tilde{H}_0(X/A) \to 0$$

where $i : A \hookrightarrow X$

$q : X \to X/A$.

$\partial = ?$

Cor. $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{otherwise.} \end{cases}$

Pf. Take $X = D^n$, $A = S^{n-1}$ $\rightsquigarrow X/A = S^n$

Induction on n.

$\tilde{H}_0(S^0) = \mathbb{Z}$.

Let $n > 0$. By Thm:

$$\cdots \to \underset{0}{\cancel{\tilde{H}_i(D^n)}} \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1}) \to \underset{0}{\cancel{\tilde{H}_{i-1}(D^n)}}$$

$$\Longrightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

Application

Brouwer Fixed Pt Thm

Any map $f: D^n \to D^n$ has a fixed pt

Pf. Assume f has no fixed pt

$\to$ retraction $r: D^n \to S^{n-1}$

retraction means:

$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$$ compos. is id

$\Longrightarrow r_* i_* = \mathrm{id}$.

but $\tilde{H}_{n-1}(D^n) = 0$ contrad. $\square$

Same as $n=2$ case with π_1 replaced by H_{n-1}.

Will prove an alternate version of the theorem first: 1′

Relative Homology

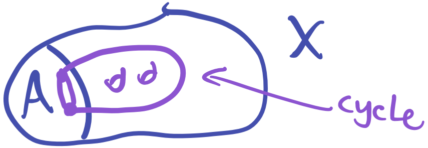
$A \subseteq X \rightsquigarrow C_n(X,A) = C_n(X)/C_n(A)$

Since ∂ takes $C_n(A)$ to $C_{n-1}(A)$ have:

$$\cdots \longrightarrow C_n(X,A) \longrightarrow C_{n-1}(X,A) \longrightarrow \cdots$$

$\rightsquigarrow$ relative homology gps $H_n(X,A)$.

Elts of $H_n(X,A)$ are relative cycles:



Will show: $H_n(X,A) \cong H_n(X/A)$

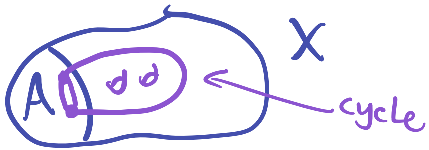
<u>Thm 1′</u>. Long ex. seq.

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \longrightarrow \cdots$$

Proof is "diagram chasing"

Thm 1'. Long ex. seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \to H_{n-1}(A) \to \cdots$$



$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_{n-1}(A) \arrow[d, "i"] \\
\tilde{c} & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_{n-1}(X) \arrow[d, "q"] \\
 & C_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d] & C_n(X,A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X,A) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

$\partial(\tilde{c})$

c cycle

0