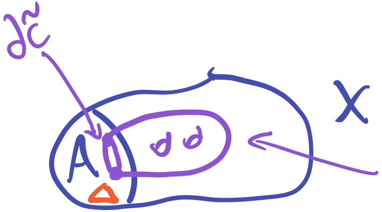
Feb 25

Thm 1'. Long ex. seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$



cycle $\tilde{c}$ (represents c (which represents $[c]$))

$[a] := \partial[c]$

"diagram chasing"

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_{n-1}(A) \arrow[d, "i"] \\
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_{n-1}(X) \arrow[d, "q"] \\
C_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d] & C_n(X,A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X,A) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

a

$\partial\tilde{c} \in \ker q$

$= i(a)$

$\tilde{c}$

c

$[c]$

0

$$\begin{tikzcd}
C_n(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_{n-1}(A) \arrow[d, "i"'] \\
C_n(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_{n-1}(X) \arrow[d, "q"'] \\
C_n(X,A) \arrow[r, "\partial"] & C_{n-1}(X,A)
\end{tikzcd}$$

a'

$[a] := \partial[c]$

$a = \partial\tilde{c}$

$a'' = a + \partial(a')$

$\tilde{c}$ $\quad$ $\tilde{c}'$

$\partial\tilde{c} \in \ker q$

$= i(a)$

c $\quad$ 0

Claim: ∂ is a well-def homom.

Two choices: • c in $[c]$

• $\tilde{c}$

We'll check that choice of $\tilde{c}$ doesn't matter.

Say $\tilde{c}'$ another choice...

Then $q(\tilde{c}) = q(\tilde{c}')$ or:

$$\tilde{c}' = \tilde{c} + i(a')$$

Instead of $\partial\tilde{c}$ we get

$$\partial\tilde{c}' = \partial\tilde{c} + \partial i(a')$$

$$= \partial\tilde{c} + i\partial(a')$$

This is homologous to a since

$$\partial(a') = 0 \text{ in } H_{n-1}(A).$$

You: check choice of c & homom.

Thm 1'. Long ex. seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Pf. More diagram chasing.
6 containments to check.

- $\operatorname{Im} \partial \subseteq \ker i_*$ i.e. $i_*\partial = 0$.
  $i_*\partial$ takes $[c]$ to $[\partial\tilde{c}] = 0$.
- $\ker i_* \subseteq \operatorname{Im} \partial$: Say $a \in C_{n-1}(A)$
  $a \in \ker i_* \Rightarrow i(a) = \partial c \quad c \in C_n(X)$
  $\Rightarrow q(c)$ is a cycle (its ∂ is in A)*
  and $\partial[q(c)] = a$

* $\partial q(b) = q\partial(b) = q(a) = 0$ $\square$

Some facts about relative hom:

Prop. $H_n(X,A) = 0 \ \forall n \Leftrightarrow H_n(A) = H_n(X) \ \forall n$

Reduced relative homology makes sense

$\rightsquigarrow \tilde{H}_n(X,A) = H_n(X,A)$ if $A \neq \emptyset$.

Prop. If $f,g : (X,A) \to (Y,B)$ homotopic then $f_* = g_*$

More: For a triple $B \subseteq A \subseteq X$

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A)$$
$$\to H_{n-1}(A,B) \to \cdots$$

Then spectral sequences...

<u>Next</u>

<u>Thm 2</u>. $A, B \subseteq X$

Mayer-Vietoris

interiors of A, B cover X

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X)$$
$$\to H_{n-1}(A \cap B) \to \cdots$$

"Van Kampen for Homology"

<u>Example</u>. $S^n = X$ $\quad A = B = D^n$

$A \cap B = S^{n-1}$

$\rightsquigarrow H_n(S^n)$.