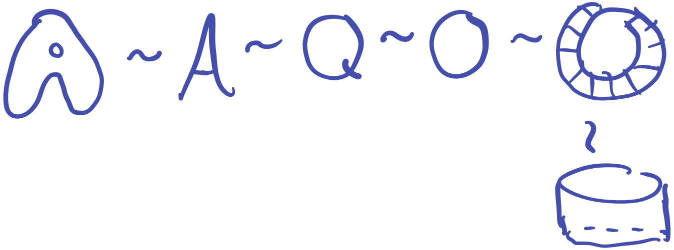
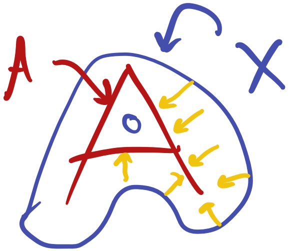
Equivalence of spaces

Jan 14

Intuition: Two spaces are same if one can be deformed to the other:





Special case. A deformation retraction $X \to A$ is a continuous family

$$\{ f_t : X \to X \mid t \in I \}$$

s.t. $f_0 = id$

$f_1(X) = A$

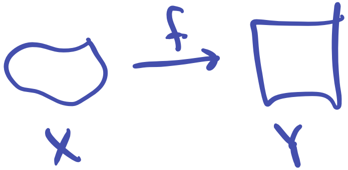
$f_t|_A = id$ for all t.

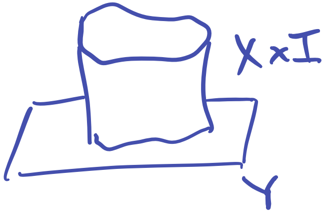
Continuous means $X \times I \to X$, $(x,t) \mapsto f_t(x)$ is continuous.

Example Given $f: X \to Y$ the mapping cylinder is

$$M_f = (X \times I) \amalg Y / \sim$$

$$(x,1) \sim f(x)$$

So: Given 

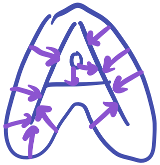
Make: 

e.g.  is M_f for

$X =$  boundary

$Y = A$

$f: X \to Y$ given by

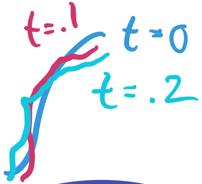


Fact. M_f def. retracts to Y

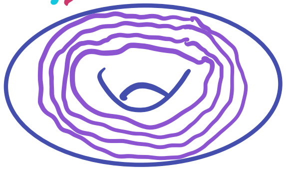
Homotopy equivalence

A *homotopy* is a continuous family $\{f_t : X \to Y : t \in I\}$

examples ① def. retr.

② 

$X = I$
$Y = \mathbb{R}^2$

③ 

$X = S^1$
$Y = T$

Say f_0, f_1 are *homotopic maps*.

A map $f: X \to Y$ is a *homotopy equivalence* if there is a $g: Y \to X$ s.t.

$$f \circ g \simeq id \simeq g \circ f$$

homotopic.

Say X, Y homotopy equivalent.

or: same homotopy type.

or: $X \simeq Y$

exercise: This is an equiv. relation.

Fact. If A is a def. retr. of X then $A \simeq X$.

Exercise:

Exercise: $\mathbb{R}^n \simeq *$ ← pt.

(def retr.)

Say: $\mathbb{R}^n$ is <u>contractible</u>.

Read: House with two rooms.

Two Criteria for Homotopy Equivalence

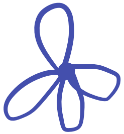
① (X, A) = CW-pair (i.e. A is a subcomplex)

A contractible

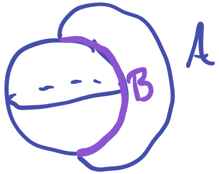
$\Longrightarrow X \simeq X/A \longleftarrow$ identify A to one pt.

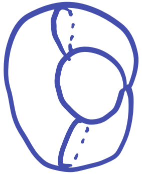
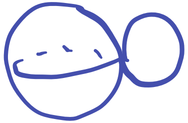
<u>example</u>. X = graph

A = any edge connecting distinct vertices.

Thus: any graph $\simeq$ 

<u>example</u>.

X = 

$X/A \simeq$  $X/B \simeq$ 

② (X, A) CW-pair.

$f, g : A \to Y$ homotopic

$\Longrightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

$X \sqcup_f Y = (X \amalg Y)/_{a \sim f(a)}$