Two Criteria for Hom. Equiv

① (X, A) CW pair
A contractible
$\Rightarrow X \simeq X/A$

② (X, A) CW pair
$f, g : A \to Y$ homotopic
$\Rightarrow X \sqcup_f Y \simeq X \sqcup_g Y$

Examples

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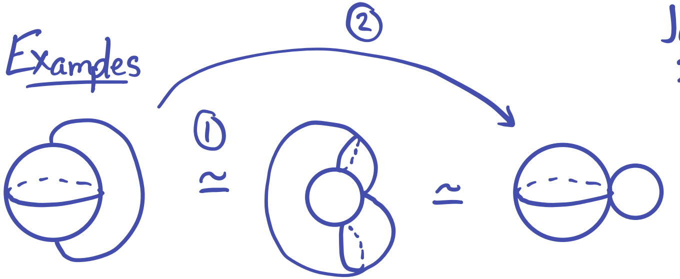
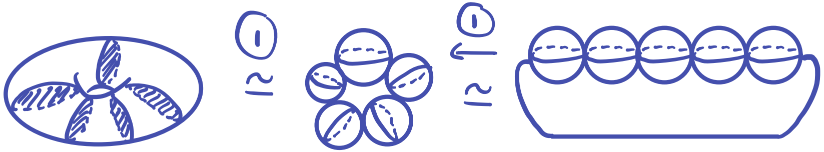
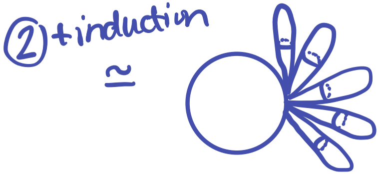
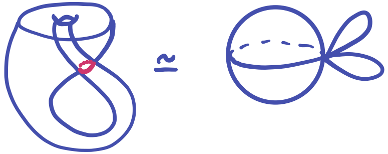






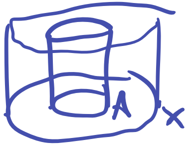
exercise. prove these equivalences using ① & ②

HW:



Both proofs use:

<u>Homotopy</u> <u>extension</u> <u>property</u>



Say (X,A) has HEP

if whenever we have

$f_0 : X \to Y$

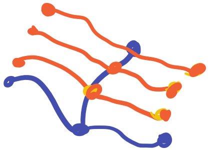
$f_t : A \to Y$

restriction of first f_0 to A is the 2^{nd} f_0

we can extend f_t to all of X.

<u>Example</u>.  X $Y = \mathbb{R}^2$

Given 

Extend 

$X \times I$  M_i

HEP same as:

Every $F : M_i \to Y$ $(i : A \hookrightarrow X)$

extends to $\tilde{F} : X \times I \to Y$

A retraction of a space X to a subspace A is a contin. map $r : X \to A$

s.t. $r|A = \mathrm{id}$.

e.g. ① $X = I$, $A = \bullet$ $r = \text{const.}$

② $X = I$, $A = :$ no retraction IVT

③ $X = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$ $A = \bullet$ $X \times I$ retracts to M_i

<u>Prop</u>. (X,A) has HEP $\Leftrightarrow$ M_i is a retract of $X \times I$

<u>Pf</u>. ($\Rightarrow$) $Y = M_i$, $F = \mathrm{id}$. $\rightsquigarrow$ $\tilde{F}$ is retract.

($\Leftarrow$) $X \times I \xrightarrow{r} M_i \xrightarrow{F} Y \rightsquigarrow \tilde{F} = F \circ r$

$\square$

Note. If X def ret to A, by f_t, then $f_1 : X \to A$ is a retraction.

Lemma. If (X,A) is a CW pair then M_i is a def. ret. of $X \times I$. In partic. (X,A) has HEP.

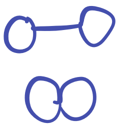
Pf. Special case. $X = D^n$, $A = \partial D^n$.

$n=2$ picture

$X \times I$ $\quad M_i$: cup shape

General case: Retract each n-cell of $X^n \setminus A^n$ to $M_i : A^n \to X^n$ during $[\frac{1}{2}^{n+1}, \frac{1}{2}^n]$

Continuous, since it is cont. on each cell. $\square$



Prop 1. (X,A) has HEP
A contractible
$\Longrightarrow q : X \to X/A$ is hom. eq.

Pf. Need a homotopy inverse

$X/A \to X$

Def Retract $A \to a$ $\quad a \in A$

Extend: $f_t : X \to X$

Since $f_1(A) = pt$, can regard f_1 as a map $X/A \to X$. $\square$

For proof of second criterion, see book or 2012 notes