Office Hours

Fri 11-12, by appt.

HW p.38 5, 6 & A1

web site

Fundamental Group

X = top. space

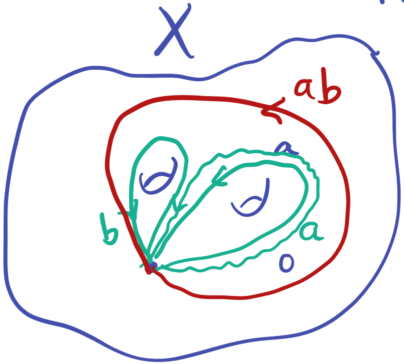
informally:

$\pi_1(X)$ = fund. gp of X

= group of based loops in X

up to homotopy

operation: concatenation.



Examples (without proof)

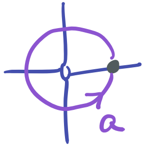
⓪ $\pi_1(\mathbb{R}^2) = 1$

① $\pi_1(S^2) = 1.$

$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ (later)

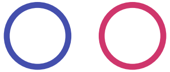
$\Longrightarrow S^2 \not\cong \mathbb{R}P^2$

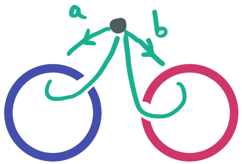
② $\pi_1(\mathbb{C} - 0) \cong \mathbb{Z}$



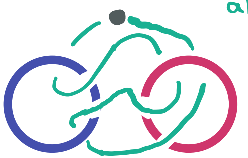
③ $\pi_1(\mathbb{R}^3 - \text{unknot}) \cong \mathbb{Z}$

④ $\pi_1(\mathbb{R}^3 - \text{unlink})$



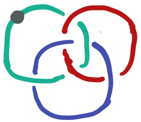


$ab = ba$? or $aba^{-1}b^{-1} = 1$?



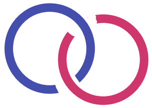
$aba^{-1}b^{-1}$

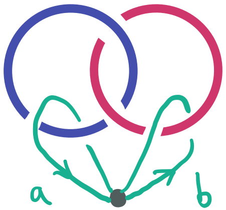
Borromean rings.



Not abelian!

⑤ $\pi_1(\mathbb{R}^3 - \text{Hopf link})$





$aba^{-1}b^{-1}$

$aba^{-1}b^{-1} = 1$

$\Longleftrightarrow ab = ba$

$aba^{-1}b^{-1}$

abelian.

Formal defns

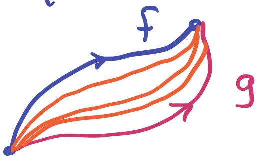
X = space

A path in X is a map $I \to X$

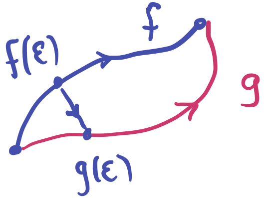
always continuous.

$[0,1]$

A homotopy of paths is a homotopy $f_t : I \to X$ s.t. $f_t(0)$ & $f_t(1)$ are indep. of t.



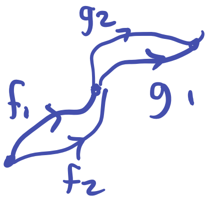
example. Any two paths f, g in $\mathbb{R}^2$ with same endpts are homotopic by straight line homotopy $f_t = (1-t)f + tg$



exercise. homotopy of paths is equiv. reln.

The composition of paths f, g with $f(1) = g(0)$ is the path

$$f g(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$



exercise. $f_1 \simeq f_2$, $g_1 \simeq g_2$ $\Longrightarrow$ $f_1 g_1 \simeq f_2 g_2$

A <u>loop</u> is a path f with $f(1) = f(0)$.

The fundamental group of X based at x_0 is

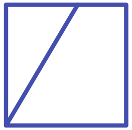
$$\pi_1(X, x_0) = \{\text{homotopy classes of loops at } x_0\}$$

group op: composition as above.

<u>Prop</u>. $\pi_1(X, x_0)$ is a group.

<u>Pf</u>. Identity: const loop.

const f

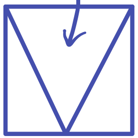


f

Note: A homotopy of paths is $I \times I \to X$

Inverses:

const



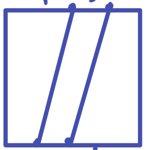
f $\bar{f}$

$\bar{f}(t) = f(1-t)$

f backwards.

Associativity

f g h



fg h

$(fg)h = f(gh)$

exercise: write details. $\square$