Fundamental groups

Jan 24

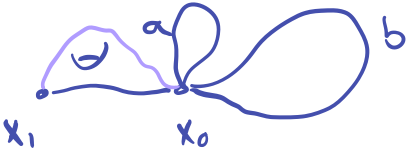
$\pi_1(X, x_0) = \{\text{loops in } X \text{ at } x_0\}/\sim$

Prop. X path conn.

$x_0, x_1 \in X$

$\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1)$

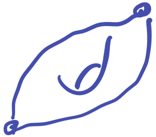
The $\cong$ given by any path x_0 to x_1



So: the $\cong$ is not canonical: different paths give diff $\cong$'s.

Say X is simply connected if $\pi_1(X) = 1$

Same as: given 2 pts, any two paths between are homotopic.

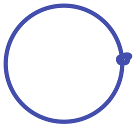


Fact. X contractible CW complex. $\Rightarrow$ X simply conn.

Use: $\exists$ strong def ret. to a pt when X is contract. CW complex (uses HEP).

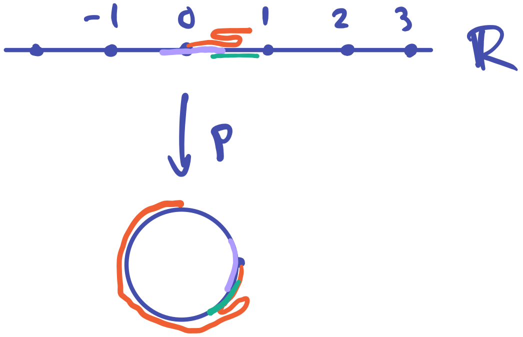
Fundamental Group of the Circle

Thm $\pi_1(S^1) \cong \mathbb{Z}$



Pf outline. Consider

$$p: \mathbb{R} \to S^1$$
$$t \mapsto e^{2\pi i t}$$



Given a loop $f: I \to S^1$

want to find a lift

$$\tilde{f}: I \to \mathbb{R} \quad \text{so}$$

$p \circ \tilde{f} = f$. & $\tilde{f}(0) = 0$.

Then the map $\pi_1(S^1) \to \mathbb{Z}$

is $\quad [f] \mapsto \tilde{f}(1)$.

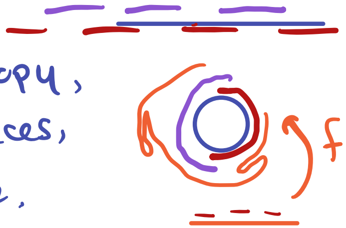
Well-def. existence/uniqueness of lifts.

Multiplicativity. easy.

Injectivity. Homotopy in $\mathbb{R}$ $\leadsto$ homotopy in S^1 via p.

Surjectivity. easy.

Remains: lifting paths / homotopies in S^1 to $\mathbb{R}$.



Idea: Given a path / homotopy, cut it into small pieces, lift them one by one.

Lemma. Given $F : Y \times I \to S^1$

$\tilde{F} : Y \times \{0\} \to \mathbb{R}$

a lift of $F|_{Y \times \{0\}}$

$\exists!\ \tilde{F} : Y \times I \to \mathbb{R}$ lifting F,

extending $\tilde{F}|_{Y \times \{0\}}$.

$F = p \circ \tilde{F}$

Path lifting: $Y = pt$.

Homotopy lifting: $Y = I$

Proof ($Y = \{y_0\}$ case)

Cover S^1 by $\{U_\alpha\}$ s.t. $\forall \alpha$ $p^{-1}(U_\alpha)$ is a disjoint union of open intervals, each homeo to U_α

F contin, I compact $\Longrightarrow$ can choose $0 = t_0 < \cdots < t_m = 1$ s.t. $\forall i$ $F([t_i, t_{i+1}]) \subset U_\alpha$ some α.

Induct: if $\tilde{F}$ defined on $[0, t_i]$ & $\tilde{F}(t_i) \in \tilde{U}_i$ use the homeo $h : \tilde{U}_i \to U_\alpha$

Define $\tilde{F}$ on $[t_i, t_{i+1}]$ via

$h^{-1} \circ F|_{[t_i, t_{i+1}]}$ $\square$

Prop. X, Y path conn.

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

Cor. $\pi_1(T^2) = \mathbb{Z}^2$.

Induced homomorphisms

$$\varphi : (X, x_0) \longrightarrow (Y, y_0)$$

$$\rightsquigarrow \varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

$$[f] \longmapsto [\varphi \circ f]$$

"Functoriality":

① $(\varphi\psi)_* = \varphi_* \psi_*$

② $id_* = id$.

Fact. φ a homeo $\Longrightarrow$ φ_* is $\cong$.

Pf. $\varphi_* \circ (\varphi^{-1})_* \overset{①}{=} (\varphi\varphi^{-1})_* = id_* \overset{②}{=} id$ $\square$