Applications of π_1

Jan 26

Brouwer's Fixed Pt thm

Every $h: D^2 \to D^2$ has a fixed pt.

Last time: $f:(X,x_0) \to (Y,y_0)$

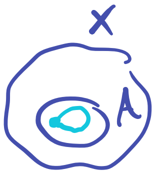
$\rightsquigarrow f_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$

Prop. $i: A \to X$ inclusion
$r: X \to A$ retraction

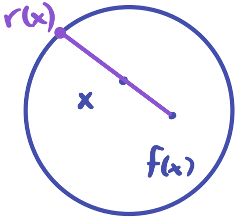
$\Longrightarrow i_*$ injective.

Pf. $r \circ i = id$

$\Rightarrow r_* \circ i_* = id$ $\square$



Pf of BFPT Suppose $h: D^2 \to D^2$ has no fixed pt. Then there is a retraction $D^2 \to S^1$



Prop $\Rightarrow$ $\pi_1(S^1) \to \pi_1(D^2)$ inj.

$$\begin{matrix} \pi_1(S^1) & \to & \pi_1(D^2) \\ \| & & \| \\ \mathbb{Z} & & 1 \end{matrix} \qquad \square$$

Also: • Borsuk Ulam Thm
any $f: S^2 \to \mathbb{R}^2$ has $f(x) = f(-x)$

• Ham sandwich thm.

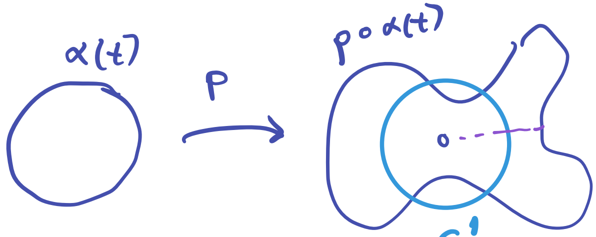
• If S^2 = union of 3 closed sets, one has a pair of antip. pts.

Fundamental Thm of Alg

Every nonconst. poly with coeff's in $\mathbb{C}$ has a root in $\mathbb{C}$.

Idea. If $\alpha(t)$ is a loop in $\mathbb{C}$ & $p(z)$ is a poly with no roots on $\alpha(t)$

$\rightsquigarrow$ loop in S^1 by projecting.



projection: $x \longmapsto x/\|x\|$.

For proof: modify α and p to get a homotopy from degree n loop in S^1 to const. loop.

Pf of Thm. Say $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$

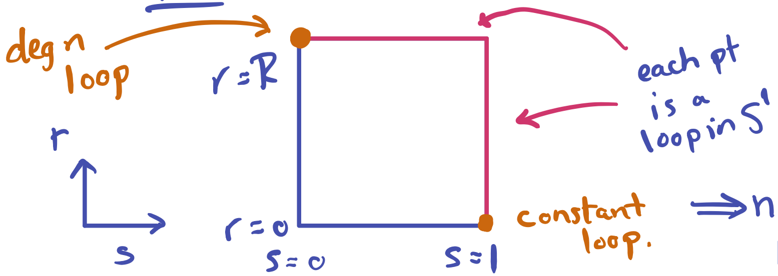
Assume no roots

Let $p_s(z) = z^n + s(a_1 z^{n-1} + \cdots + a_n)$

Let $\alpha_r(t) =$ circle of radius r in $\mathbb{C}$.

Let $R > |a_1| + \cdots + |a_n| + 1$

Claim. $p_s(z)$ has no roots on α_R



$\Rightarrow n = 0$

$\square$

Last time:

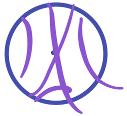
Fact. f homeo $\Longrightarrow$ f_* isom.

Prop. $\pi_1(S^n) = 1$, $n > 1$

Pf. $S^n - pt \underset{\text{homeo}}{\cong} \mathbb{R}^n$

(stereographic proj).

Suffices to show: any loop is homotopic to one that is not surjective.



Apply above fact about f_* $\square$

Prop. $\mathbb{R}^2 \not\cong \mathbb{R}^n$ $n > 2$

Pf. $\mathbb{R}^n - \underset{pt}{\overset{\text{any}}{}} \cong S^{n-1} \times \mathbb{R}$

(polar coords)

By Fact

$$\begin{aligned}\pi_1(\mathbb{R}^n - pt) &\cong \pi_1(S^{n-1} \times \mathbb{R}) \\ &\cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \\ &\cong \pi_1(S^{n-1}) \\ &\cong \begin{cases} \mathbb{Z} & n = 2 \\ 1 & n > 2 \end{cases}.\end{aligned}$$

Apply Fact. and the "any" $\square$

Prop. If $\varphi : (X, x_0) \to (Y, y_0)$ hom eq.

then φ_* is $\cong$.

Pf. Let ψ be homotopy inverse

i.e. $\psi \circ \varphi \simeq id$.

Remains to show: $(\psi \circ \varphi)_* = \cancel{id_* = id.}$ isomorphism.

Or: $H_t : X \to X$ homotopy

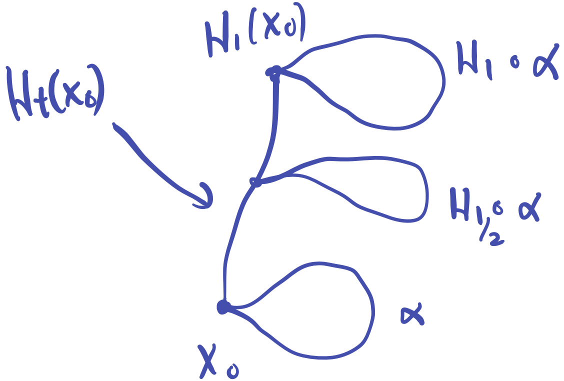
$H_0 = id$

Then $(H_1)_* : \pi_1(X, x_0) \to \pi_1(X, H_1(x_0))$

is an isomorphism.

Proof of this last fact:

It's the change of basept isomorphism from last time.



$\square$