Jan 28

Goal: compute π_1 for lots of spaces

Free groups and free products

$F_n = \{$freely reduced words in $x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$

group op: concatenation, free reduce.

existence of F_n is nontrivial! (associativity)

G, H groups

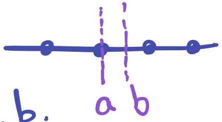
$G * H$ = free product of G & H

= $\{$freely reduced words in $G, H\}$

elts look like $g_1 h_1 g_2 h_2 \cdots g_n h_n$

or $g_1 h_1 \cdots g_n$ etc.

Examples ① $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$

= symmetries of 

= alt words in a, b.

② $\mathbb{Z} * \mathbb{Z} \cong F_2$

Properties ① $G, H \leq G * H$

② $G \cap H = 1$

③ Given $G \to K$, $H \to K$ group.

$\exists!$ $G * H \to K$

"univ. property"

Van Kampen's Thm

$X = A \cup B$ A, B open, path conn.
$A \cap B$ path conn.

$x_0 \in A \cap B$ basept for $X, A, B, A \cap B$

The inclusions $A, B \hookrightarrow X$ induce

$$\pi_1(A), \pi_1(B) \longrightarrow \pi_1(X)$$

By ③ on last slide we get

$$\Phi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$$

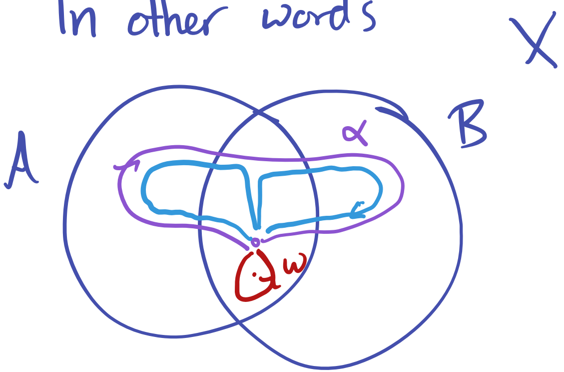
Let N be normal subgp of $\pi_1(A) * \pi_1(B)$ gen by

$i_A : A \cap B \to A$
$i_B : A \cap B \to B$

$$\big((i_A)_*(\omega)(i_B)_*(\omega)^{-1}\big) \quad \forall\, \omega \in \pi_1(A \cap B)$$

Then: ① Φ surjective
② $\ker \Phi = N$.

In other words

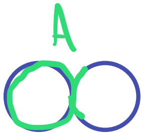


① Any $\alpha \in \pi_1(X)$ is product of loops in A, B.

② If $\omega \in \pi_1(A \cap B)$, the corresp. elts of $\pi_1(A), \pi_1(B)$ are equal.

Or: $\pi_1(X) = \pi_1(A) * \pi_1(B) / N$

Examples



① $\pi_1(S^1 \vee S^1)$

$\pi_1(A) * \pi_1(B) / N \cong \mathbb{Z} * \mathbb{Z} / 1$

$\cong F_2$

Induction: $\pi_1(\bigvee_n S^1) \cong F_n$.

$\Longrightarrow \pi_1(\mathbb{R}^2 - n \text{ pts}) \cong F_n$

$\pi_1(\mathbb{R}^3 - \text{unlink}) \cong F_n$.

$\pi_1(\text{any graph}) \cong F_n$ some n.

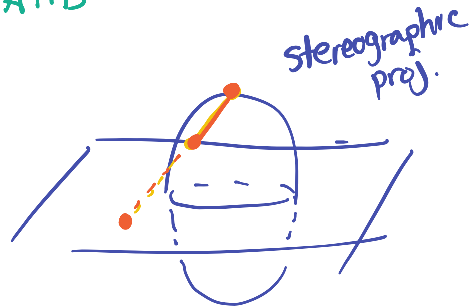
② $\pi_1(S^n) \cong 1$

$A = S^n - \text{north pole}$

$B = S^n - \text{south pole}$.

$A \cap B \simeq S^{n-1}$.

For $n=2$: $1 * 1 / 1 = 1$



③ $\pi_1(S^3 - (p,q) \text{ torus knot})$

$\cong \langle x, y : x^p = y^q \rangle$

read in Hatcher

Van Kampen in terms of presentations

$\pi_1(A) = \langle S_1 \mid R_1 \rangle$

$\pi_1(B) = \langle S_2 \mid R_2 \rangle$

$\pi_1(A) * \pi_1(B) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \rangle$

Choose a gen set S_3 for $\pi_1(A \cap B)$.

For each $\omega \in S_3$ write it as a product w_1 of elts of S_1 & as a product w_2 of elts of S_2

Let R_3 be the set of relators $w_1 w_2^{-1}$ constructed in this way.

Then:

$$\pi_1(X) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \amalg R_3 \rangle$$

Preview of next time:

Gluing a disk to X

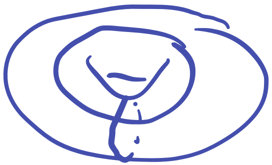
$\rightsquigarrow$ adding a relation to $\pi_1(X)$

examples ① $X = S^1$

$\pi_1(X) = \langle a \mid \ \rangle \cong \mathbb{Z}$

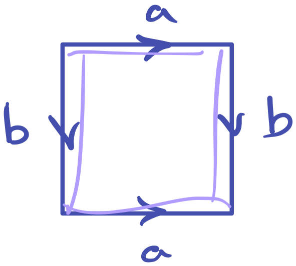
adding disk: $\pi_1(\hat{X}) = \langle a \mid a \rangle = 1$

②



$$X = S^1 \vee S^1$$

$$\pi_1(X) \cong F_2 = \langle a, b \mid \ \rangle$$



$$\pi_1(\hat{X}) = \langle a, b \mid aba^{-1}b^{-1} \rangle$$
$$= \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2.$$