Jan 31

Basics of group presentations

$G = \langle S \mid R \rangle$ means:

1. G is gen by S
2. Two words in $S^{\pm 1}$ are equal in G iff they differ by finitely many elts of R.

Fact. If $G = \langle S \mid R \rangle$ then

$$G \cong F(S) / \langle\langle R \rangle\rangle$$

Here: each elt of R is a relator like $aba^{-1}b^{-1}$ instead of a relation $ab = ba$.

Examples

$F_2 = \langle a, b \mid \ \rangle$

$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

$\mathbb{Z}/n = \langle a \mid a^n = 1 \rangle$

Check: modding out by $aba^{-1}b^{-1}$ is same as saying $ab = ba$

$$\begin{aligned} ba &= ba\left((ba)^{-1}(aba^{-1}b^{-1})(ba)\right) \\ &= ab \end{aligned}$$

Van Kampen's Thm

$X = A \cup B$ A, B open, path conn.
$A \cap B$ path conn.

$x_0 \in A \cap B$ basept for $X, A, B, A \cap B$

The inclusions $A, B \hookrightarrow X$ induce

$$\pi_1(A),\ \pi_1(B) \longrightarrow \pi_1(X)$$

By ③ ~~on last slide~~ we get

$$\Phi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$$

Let N be normal subgp of $\pi_1(A) * \pi_1(B)$ gen by

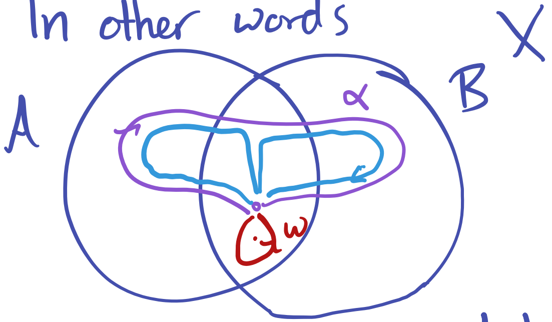
$$\big((i_A)_*(w)\big)\big((i_B)_*(w)\big)^{-1} \quad \forall\, w \in \pi_1(A \cap B)$$

$i_A : A \cap B \to A$
$i_B : A \cap B \to B$

Then: ① Φ surjective
② $\ker \Phi = N$.

Or: $\pi_1(X) = \pi_1(A) * \pi_1(B) \,/\, N$

In other words



① Any $\alpha \in \pi_1(X)$ is product of loops in A, B.

② If $w \in \pi_1(A \cap B)$, the corresp. elts of $\pi_1(A), \pi_1(B)$ are equal.

and: all relations come from this & those in A, B

Van Kampen in terms of presentations

$\pi_1(A) = \langle S_1 \mid R_1 \rangle$, $\pi_1(B) = \langle S_2 \mid R_2 \rangle$

$\Rightarrow \pi_1(A) * \pi_1(B) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \rangle$

Choose a gen set S_3 for $\pi_1(A \cap B)$.

For each $\omega \in S_3$ write it as a product w_1 of elts of S_1 & as a product w_2 of elts of S_2

Let R_3 be the set of relations $w_1 = w_2$ constructed in this way.

Then:

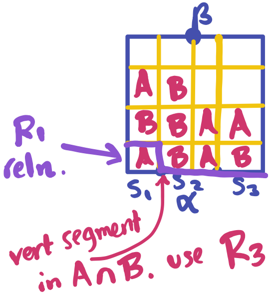
$$\pi_1(X) = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \amalg R_3 \rangle$$

Proof of VKT

① is compactness of I plus above picture.

② is... Say α, β are words in $S_1 \amalg S_2$ that are equal in $\pi_1(X)$. Must show they differ by $R_1 \amalg R_2 \amalg R_3$

Since $\alpha = \beta$ in $\pi_1 X$, get homotopy.



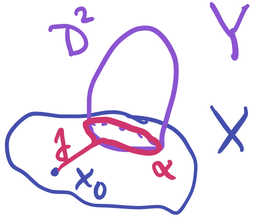
By compactness, chop into smaller squares, each mapping to A or B.

Push across one at a time

Contemplate.

Attaching Disks

X path conn., based at x_0



Attach 2-cell D^2 via

$$\varphi : S^1 \to X \rightsquigarrow Y = X \cup \text{disk}.$$

Choose path γ from x_0 to $\varphi(S^1)$

The loop $\overset{\alpha}{}\ \gamma\varphi(S^1)\bar{\gamma}$ is null-hom. in Y.

Let $N = \langle\langle \cancel{\text{this loop}}\ \alpha \rangle\rangle$

Prop. Inclusion $X \hookrightarrow Y$ induces $\pi_1(X) \twoheadrightarrow \pi_1(Y)$ with kernel N.

So: $\pi_1(Y) \cong \pi_1(X)/N$.

Proof. Van Kampen.

$x \in D^2$. $A = Y - x$.

$B = \operatorname{int} D^2$

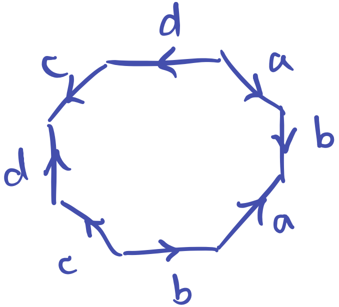
$$A \cap B = \operatorname{int} D^2 - x \simeq S^1$$

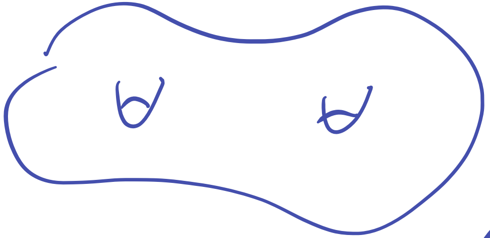
$$\text{VKT} \Rightarrow \pi_1(Y) = \pi_1(X) * \pi_1(D^2) \big/ N$$

$$\cong \pi_1(X)/N$$

where $N = \langle\langle \alpha \cdot 1^{-1} \rangle\rangle$ $\square$

Examples

① $M_2 =$ 

$=$ 

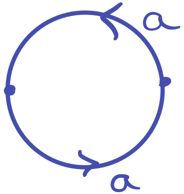
$\simeq \bigvee_4 S^1 \cup \text{disk}$

$aba^{-1}b^{-1}$

$\rightsquigarrow \pi_1(M_2) = F_4/\langle\langle \partial D^2 \rangle\rangle = \langle a,b,c,d \mid [a,b][c,d] = 1 \rangle$

Similar: $\pi_1(M_g) = \langle x_1, y_1, \dots, x_g, y_g \mid [x_1,y_1]\cdots[x_g,y_g] = 1 \rangle$

Conseq. $\pi_1(M_g)^{ab} = \mathbb{Z}^{2g} \implies$ If $g \neq h$ then $M_g \not\simeq M_h$.

② $\mathbb{RP}^2 =$ 

$\implies \pi_1(\mathbb{RP}^2) = F_1/\langle\langle a^2 \rangle\rangle \cong \mathbb{Z}/2$.

So: $\mathbb{RP}^2 \not\simeq S^2$