Applications of Homology

Mar 4

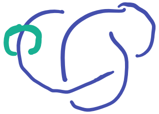
① Jordan curve thm.

Thm. Let $h: S^1 \to \mathbb{R}^2$ be an embedding (injective). Then $\mathbb{R}^2 \setminus h(S^1)$ has exactly 2 connected components.

Osgood curves...

(b) also implies

$$H_1(S^3 \setminus \text{knot}) = \mathbb{Z}$$



Prop. (a) If $h: D^k \to S^n$ is an embedding then

$$\tilde{H}_i(S_n \setminus h(D^k)) = 0 \quad \forall i.$$

(b) If $S^k \to S^n$ is embedd. $k<n$

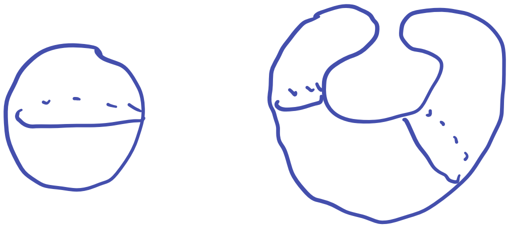
$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{o.w.} \end{cases}$$

(b) implies any S^{n-1} in S^n divides S^n into 2 components, each with the homology of a pt.

For $n=2$: Jordan curve

For $n=3$: It is possible for one component to not be simply conn.

Alexander Horned Spheres



etc.

Outside has π_1 ∞ gen. but by Prop. $H_1 = 0$.

$$\pi_1(\text{outside}) = \langle \alpha_0, \alpha_1, \ldots \mid [\alpha_1, \alpha_2] = \alpha_0,\ [\alpha_3, \alpha_4] = \alpha_1,\ [\alpha_5, \alpha_6] = \alpha_2, \ldots \rangle$$

etc.

intersect these.

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$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{o.w.} \end{cases}$$

ⓐ Induct on k.

$k=0$: $S^n - h(D^k) \cong \mathbb{R}^n$ ✓

Replace D^k with I^k

Let $A = S^n - h(I^{k-1} \times [0, 1/2])$ ← half-cube

$B = S^n - h(I^{k-1} \times [1/2, 1])$

$A \cup B = S^n - h(I^{k-1} \times \{1/2\})$

Induction $\Longrightarrow \tilde{H}_i(A \cup B) = 0$.

Mayer-Vietoris

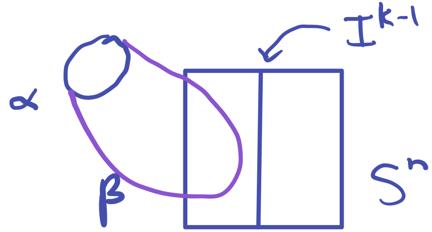
$\Phi : \tilde{H}_i(A \cap B) \xrightarrow{\cong} \tilde{H}_i(A) \oplus \tilde{H}_i(B)$

($A \cap B = S^n - h(I^k)$)

Assume for contrad $[\alpha] \neq 0$ in $\tilde{H}_i(A \cap B)$

Then $[\alpha] \neq 0$ in $\tilde{H}_i(A)$ (Φ of) $A = S^n -$ half cube.

But $\alpha = 0$ in $\tilde{H}_i(A \cap B) = 0$. repeat!



$\Rightarrow \alpha = \partial\beta$

$\beta \subseteq S^n - I^{k-1}$

By compactness, β lives in some finite stage.

Prop. (a) If $h: D^k \to S^n$ is an embedding then

$$\tilde{H}_i(S_n \setminus h(D^k)) = 0 \quad \forall i,$$

(b) If $S^k \to S^n$ is embedd. $k<n$

$$\tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{o.w.} \end{cases}$$

Proof of (b) Induct on k.
& MV.

Exercise. The case $k=n$.

$\rightsquigarrow$ S^n cannot embed in $\mathbb{R}^n$

$\mathbb{R}^m$ cannot embed in $\mathbb{R}^n$ $m>n$.

(2) Invariance of domain

Thm. U open in $\mathbb{R}^n$

$h: U \to \mathbb{R}^n$ embedding

$\Longrightarrow h(U)$ open in $\mathbb{R}^n$.

Cor. M = compact n-manifold

N = connected n-man

Any embedding $M \to N$ is surjective, hence a homeo.

Thm. U open in $\mathbb{R}^n$

$h: U \to \mathbb{R}^n$ embedding

$\Rightarrow h(U)$ open in $\mathbb{R}^n$.

Pf. Think of $\mathbb{R}^n$ as $S^n \setminus pt$

Enough to show $h(U)$ open in S^n

Let $x \in U$, D^n = disk about x in U

Suffices to show $h(\text{int } D^n)$ open in S^n.

Prop (b) $\Rightarrow$ $S^n \setminus h(\partial D^n)$ has 2 path comp.

The two path components are: (you: justify)

$h(\text{int } D^n)$

$S^n \setminus h(D^n)$

$h(\partial D^n)$ is closed (compact in Hausdorff)

$\Rightarrow S^n \setminus h(\partial D^n)$ open.

$\Rightarrow$ path components are the conn. components. (true in a loc. comp. space)

Fact. An open set with finitely many components has each component as an open subspace.

$\Rightarrow h(\text{int } D^n)$ open in $S^n \setminus h(\partial D^n)$

$\Rightarrow$ open in S^n. $\square$