Mar 7

② Invariance of domain

Thm. U open in $\mathbb{R}^n$

$h: U \to \mathbb{R}^n$ embedding

$\Rightarrow h(U)$ open in $\mathbb{R}^n$.

Cor. M = compact n-manifold*

N = connected n-man

Any embedding $M \to N$

is surjective, hence a homeo.

every pt has a nbd homeo to $\mathbb{R}^n$. + Haus.

Pf of Cor.

$h(M)$ closed in N (compact in Haus.)

Since N connected, suffices to show

$h(M)$ open in N.

Let $x \in M$, Choose nbd $V \cong \mathbb{R}^n$ of $h(x) \in N$.

Choose open nbd U of x in $\overset{\text{open}}{\overset{\shortparallel}{h^{-1}(V)}}$

homeo to $\mathbb{R}^n$.

But $h|U$ is an embedding (restr. of embed)

$\Rightarrow h(U)$ open by Thm. $\Rightarrow h(M)$ open

in V, hence N. $\square$

③ Division Algebras

An algebra over $\mathbb{R}$ is $\mathbb{R}^n$ with bilin. mult.

$a(b+c) = ab + ac$ etc...

It's a division alg if $ax = b$ always solvable for $a \neq 0$ (no zero divisors).

Examples. $\mathbb{R}$, $\mathbb{C}$

Thm. $\mathbb{R}, \mathbb{C}$ are only finite dim algebras over $\mathbb{R}$ that are commutative and have id.

Pf that any such alg has dim ≤ 2:

Define $f: S^{n-1} \to S^{n-1}$ by $f(x) = x^2/|x^2|$

$\rightsquigarrow$ induced map $\bar{f}: \mathbb{R}P^{n-1} \to S^{n-1}$

well def: no 0 divisors!

Claim $\bar{f}$ injective.

$\alpha = \left(\frac{|x^2|}{|y^2|}\right)^{1/2}$

Pf $\bar{f}(x) = \bar{f}(y) \Rightarrow x^2 = \alpha y^2$

$\Rightarrow x^2 - \alpha^2 y^2 = 0 \overset{\text{commut!}}{\Longrightarrow} (x + \alpha y)(x - \alpha y) = 0$

$\overset{\text{no 0 div}}{\Longrightarrow} x = \pm \alpha y \Rightarrow x = y$ in $\mathbb{R}P^{n-1}$ $\square$

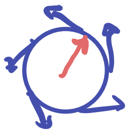
$\bar{f}$ inj on compact Haus. $\to$ embedding.

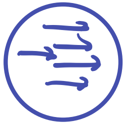
Cor $\Rightarrow$ $\bar{f}$ homeo. $\Rightarrow n \leq 2$. (use π_1 or H_1 or H_{n-1})

Some more algebra to finish the thm. $\square$

④ Hairy ball thm (Can't comb a monkey)

<u>Thm</u>. S^n has a continuous field of nowhere 0 tangent vectors iff n odd.





n odd: $v(x_1, \dots, x_{2k})$

$= (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$

<u>Tool: Degree</u>

$$f: S^n \to S^n \rightsquigarrow f_*: \underbrace{H_n(S^n)}_{\text{" } \mathbb{Z}} \to H_n(S^n)$$

$$\alpha \longmapsto d\alpha$$

d = degree of f.

<u>Facts</u>. (i) $\deg \mathrm{id} = 1$

(ii) $\deg f = 0$ if f not surj.

(iii) $\deg f = \deg g \Leftrightarrow f \simeq g$.

$\Rightarrow$ Hopf.

(iv) $\deg fg = \deg f \deg g$.

(v) $\deg f = -1$ for f a reflection thru equator.

(vi) $\deg(\text{antipodal}) = (-1)^n$.

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Pf. ($\Rightarrow$) Let $v(x)$ = vect. field on S^n.

$\rightsquigarrow$ $v(x) \perp x$ in $\mathbb{R}^{n+1}$.

$v(x) \neq 0 \ \forall x \Rightarrow$ can replace $v(x)$ with $v(x)/|v(x)|$

$\Rightarrow (\cos t)x + (\sin t)v(x)$ is a unit $S^1 \subseteq S^n$ in $x v(x)$ plane

is a homotopy from id $(t=0)$ to antipodal map $(t=\pi)$

(iii) $\Rightarrow$ $\deg \mathrm{id} = \deg$ antip.

(vi) $\Rightarrow$ n odd. $\square$

One more fact

(vi) If f has no fixed pts

$\deg f = (-1)^{n+1}$

proof: find homotopy to antip. map.
(straight line)

5 **Prop.** $\mathbb{Z}/2$ is only gp that acts freely on S^n if n even.

Pf. Say $G \curvearrowright S^n$ 〃 $\mathbb{Z}/2$

$\rightsquigarrow d : G \to \{\pm 1\}$ homom by (iv).

Action free $\Rightarrow$ $\underset{\text{by (vi)}}{d(g) = (-1)^{n+1}}$ $\forall g \neq id$.

n even $\rightarrow \ker d = 1$

$\Rightarrow G \cong \mathbb{Z}/2$ or 1

$\square$