Mar 9

6 Borsuk-Ulam Thm

Thm. $g: S^n \to \mathbb{R}^n$

$\Rightarrow \exists\ x$ s.t. $g(x) = g(-x)$

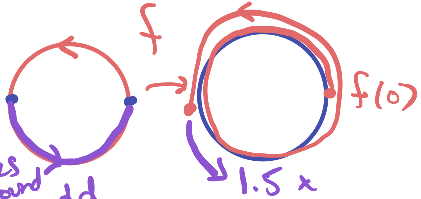
Prop. $f: S^n \to S^n$ odd $\quad f(-x) = -f(x)$

$\Rightarrow \deg f$ odd

Pf of Prop $(n=1)$

WLOG $f(0) = 0$

$\Rightarrow f(\pi) = \pi$



$f([0,\pi])$ goes $k/2$ times around, k odd

vs. $f([\pi, 0])$ goes $k/2$ times around in same dir. $\square$

Pf of Prop for $n>1$ uses $\mathbb{Z}/2$ coeffs & transfer homoms.

Pf of Thm. Let $f(x) = g(x) - g(-x)$ odd!

Say $f(x) \neq 0\ \forall x$.

Then can define $h(x) = f(x)/|f(x)|$

$h: S^n \to S^{n-1}$

$h|\text{equator}: S^{n-1} \to S^{n-1}$ still odd

Prop $\Rightarrow h|eq$ odd degree.

But $\deg h|eq = 0$

(it is $\simeq$ const since eq $\simeq$ pt in S^n) $\square$

⑦ Lefschetz Fixed Pt Thm

For $\varphi : A \to A$ $\quad A =$ fin gen abel gp

$$\operatorname{tr}\varphi = \operatorname{tr}\left(A/\text{torsion} \longrightarrow A/\text{torsion}\right)$$
$$= \operatorname{tr}\left(\mathbb{Z}^k \longrightarrow \mathbb{Z}^k\right)$$

X = space with fin gen homology

e.g. finite CW complex. of dim n

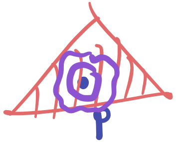
$f : X \to X$

The Lefschetz # of f is

$$\tau(f) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}\left(f_* : H_i(X) \to H_i(X)\right)$$

If $f(p) = p$ (fixed pt)

deg p is the degree of



$\bar{f} : H_n(X, X-p) \longrightarrow H_n(X, X-p)$

example: ① If f rotates about p

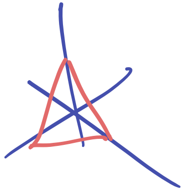
deg $p = 1$

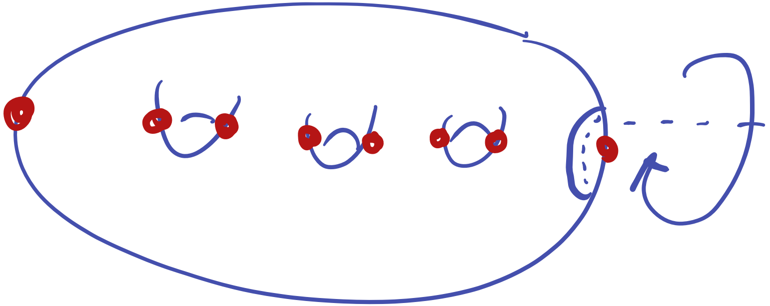
② $f(z) = z^2$ $\quad X = \mathbb{C}$ $\quad \deg 0 = 2$.

Thm. $\displaystyle \tau(f) = \sum_{f(p)=p} \deg(p)$

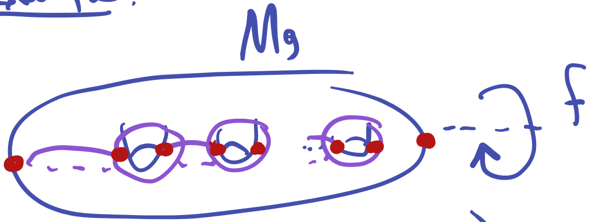
Cor. Brouwer FPT. $\tau(f) = 1$

Thm. $\tau(f) = \sum_{f(p)=p} \deg(p)$





Example.

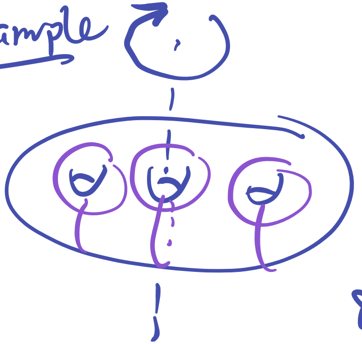


$$tr\left(f_* : H_0(M_g) \to H_0(M_g)\right) = 1$$
$$tr\left(f_* : H_1(M_g) \to H_1(M_g)\right) = -2g$$
$$tr\left(f_* : H_2(M_g) \to H_2(M_g)\right) = 1$$

$\tau(f) = 2g+2.$

There are $2g+2$ fixed pts, all of deg 1

Example



$\tau(f) = 1-2+1$
$= 0$

& no fixed pts.

Cor. Any linear map $L: \mathbb{R}^n \to \mathbb{R}^n$ n odd has a real eigenvector.

Pf. L induces $\mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$. $\tau(f)=1$ Lefschetz $\Rightarrow$ fixed pt

⑧ Euler characteristic.

X = CW complex

c_n = # n cells

$$\chi(X) = \sum (-1)^i c_i$$

Thm. This is indep of cell decomp.

Even better:

$$\chi(X) = \sum (-1)^i \underbrace{\operatorname{rk} H_i(X)}_{\text{rank of } H_i(X)/\text{torsion}}$$

C_n = usual simplicial/cellular chain complex.

Thm 2.44 in Hatcher.

Example. $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$