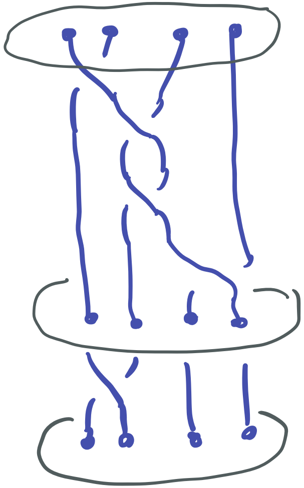
# Braid Groups

Mar 11

Artin 1925, Hurwitz 1891

$B_n = \{\text{braids on } n \text{ strands}\}/\text{homotopy}$



$\mapsto (3\,4) \in S_4$

Multiplication: stacking.

Identity: $||||||$

Inverse: Mirror vertically.

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Generators:

$\sigma_i$


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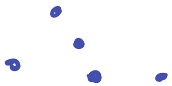
A map $B_n \twoheadrightarrow S_n$.

Kernel: $PB_n$ pure braid group.

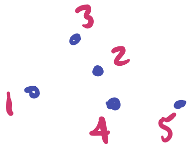
## 2nd interpretation

$C_n$ = config sp. of $n$ distinct pts in $\mathbb{R}^2$



a pt in $C_5$

$U_n$ = config sp. of $n$ labeled distinct pts in $\mathbb{R}^2$



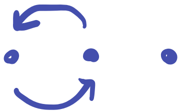

a pt in $U_5$

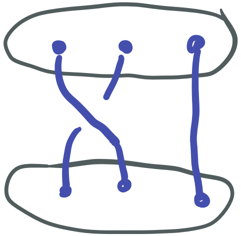
$U_n = (\mathbb{R}^2)^n \setminus$ big diag.

$\pi_1(C_n) \cong B_n$

$\pi_1(U_n) \cong PB_n$.

A loop in $C_3$:



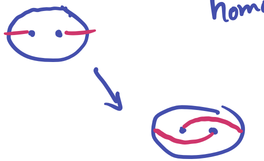


time

## 3rd interp

$B_n \cong$ Homeo $(D_n, \partial D_n)$ / homotopy

id on $\partial$



So $B_n$ encodes motions of pts (robotics, physics...)

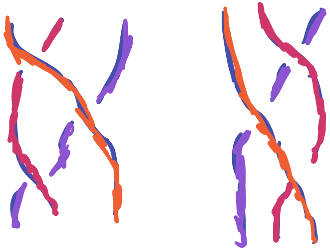
$C_n$ = Poly$_n$ = space of deg $n$ square free polynomials

## Some braid gps

$B_1 \cong 1$

$B_2 \cong \mathbb{Z}$

$B_3$ more complicated.



braid relation

$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

$SL_2\mathbb{Z}$ = $2\times 2$ $\mathbb{Z}$ matrices of det 1

but $B_3/\text{center} \cong PSL_2\mathbb{Z}$

A $K(G,1)$ space is a space with $\pi_1 = G$ & contractible univ. cover

$B_n$ $n>3$ even more complicated...

**Thm.** $B_n$ is torsion free.

(Only elt of finite order is id).

b

$b^3$ is central in $B_3$.

(exercise).

**Pf outline.** ① $C_n$ is a $2n$-dim $K(B_n,1)$

② If $G$ has torsion, any $K(G,1)$ is $\infty$ dim.

① $C_n$ is a $K(B_n, 1)$

First $U_n \to C_n$ is the covering space. corr to $PB_n$.

To see this: $S_n \curvearrowright U_n$.

$$U_n / S_n = C_n$$

This is a covering sp action.

$\odot_x \in U_n$ (exercise)

So: $U_n$ & $C_n$ have same univ cover.

So: Will show $\tilde{U}_n$ is contractible.

<u>Whitehead's Thm</u> $X$ CW complex

Then $\tilde{X}$ contractible iff $\pi_i(X) = 0$ $i > 1$

preim. of pt.

We have:

$$\begin{tikzcd} \mathbb{R}^2 - \{n-1 \text{ pts}\} \arrow[r] \arrow[d, phantom, "\shortparallel"] & U_n \arrow[d] \\ \text{graph} \cong X_{n-1} & U_{n-1} \end{tikzcd}$$

fiber bundle.

$\rightsquigarrow$ LES for fiber bundles:

$$\underset{0}{\pi_i(X_{n-1})} \to \pi_i(U_n) \to \underset{0 \text{ by ind.}}{\pi_i(U_{n-1})} \to \pi_{i-1}(X_{n-1})\ 0$$

$$\Rightarrow \pi_i(U_n) = 0. \quad \square$$

② $G$ has torsion $\Longrightarrow$ Any $K(G,1)$ is $\infty$ dim.

We'll show any $K(\mathbb{Z}/2,1)$ is $\infty$-dim

Our fave $K(\mathbb{Z}/2,1)$ : $S^\infty/(\mathbb{Z}/2) = \mathbb{RP}^\infty$.

The chain complex for $H_*(\mathbb{RP}^\infty)$

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \cdots$$

$H_n = \mathbb{Z}/2$ $\quad H_{n-1} = 0$.

$\Longrightarrow$ $\mathbb{RP}^\infty$ has nontrivial homology in $\infty$ many dim.

$\Longrightarrow$ any $K(\mathbb{Z}/2,1)$ is $\infty$ dim.

Similar for $\mathbb{Z}/n$

If $G$ has torsion then $K(\mathbb{Z}/n,1)$ is a cover of $K(G,1)$

If $K(G,1)$ finite dim then $K(\mathbb{Z}/n,1)$ fin dim contrad.

# Idea of Whitehead's thm

Pf (⇒)

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array}$$

$\tilde{X}$ contractible.

contract it!

use lifting crit!

$\pi_1 = 1$

$\longrightarrow X$

push down the contraction!

need to show homotopic to pt.

(⇐) lift all contractions of spheres + $\pi_1(\tilde{X}) = 1$ by defn.

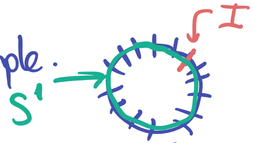
# Fiber bundles

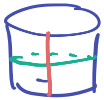
$$\begin{array}{ccc} \text{fiber } F & \longrightarrow & E \text{ total space} \\ & & \downarrow \pi \\ & & B \text{ base space} \end{array}$$

all $\pi^{-1}(b)$ homeo to $F$

and $\forall\, b \in B\ \exists$ nbd $U$ s.t. $\pi^{-1}(U) \cong U \times F$ "locally a product"

example.



$$\begin{array}{ccc} I & \longrightarrow & A \\ & & \downarrow \\ & & S^1 \end{array} \qquad \begin{array}{ccc} I & \longrightarrow & \text{Mobius} \\ & & \downarrow \\ & & S^1 \end{array}$$

$A = S^1 \times [-1,1]$

$\pi$ = proj to $S^1 \times \{0\}$

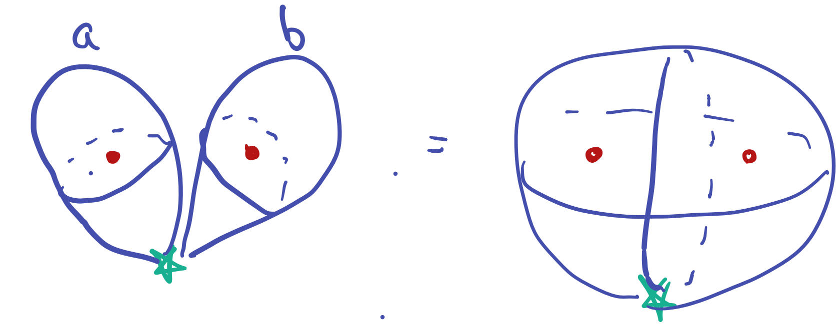
Example Cov. sp $F$ = discrete set

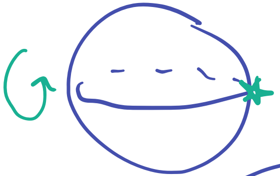
# Homotopy gps

$X = \mathbb{R}^3 - $ two pts

$\pi_1(X) = 1$

$\pi_2(X) = \mathbb{Z}^2$



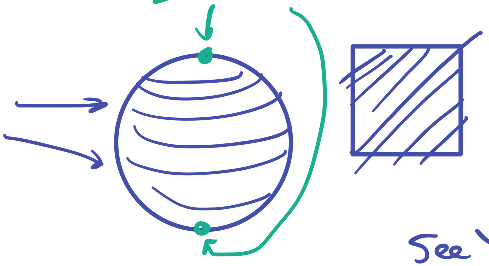



$\pi_i$ abelian

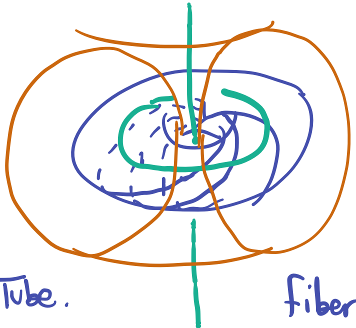
$i > 1$

Hopf fibration is an interesting elt of $\pi_3(S^2)$

tori




See YouTube.



Fill $S^3$ by tori

Fill each torus with $(1,1)$ curves

$S^3 \longrightarrow$ set of circles $= S^2$

Fiber $S^1$