Mar 14

Cohomology

Same basic info as homology but:

- mult. structure.
- pairing with homology
- contravariance.

Quick idea: $X = \Delta$ complex

$G = \mathbb{Z}$ or $\mathbb{Z}/2$ or another abel. gp.

$\Delta^i(X) =$ functions from i-simplices of X to G.

$=$ homoms $\Delta_i(X) \to G$

$$\delta : \Delta^i(X,G) \longrightarrow \Delta^{i+1}(X,G)$$
$$f \mapsto \delta f$$

for $f \in \Delta^i$, $\sigma = (i+1)$-simplex

$$\delta f(\sigma) = \sum (-1)^k f(\partial_k \sigma)$$

$H^*(X,G) =$ homology of this chain complex.

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Graphs. X = 1-dim Δ-complex = oriented graph.

Let $f \in \Delta^0(X,G)$

$\delta f(e) = f(\text{end of } e) - f(\text{start of } e)$

= change of f on e "derivative"

Think of f as elevation.

$\rightsquigarrow$ chain complex

$$0 \to \Delta^0(X,G) \xrightarrow{\delta} \Delta^1(X,G) \to 0$$

$H^0(X,G) = \ker \delta$ = constant fns on each component

= direct product of components

(vs. direct sum, like in H_0 case)

$= \prod_{\text{components of } X} G$.

$H^1(X,G) = \Delta^1(X,G)/\operatorname{Im}\delta$.

For $f \in \Delta^1(X,G)$

$[f] = 0 \Leftrightarrow f \in \operatorname{Im}\delta \Leftrightarrow f$ is an antideriv.

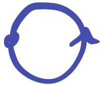
$H^1(X,G) = \Delta^1(X,G)/\operatorname{Im}\delta$. $\quad f \in \Delta^1(X,G)$

$[f] = 0 \Leftrightarrow f \in \operatorname{Im}\delta \Leftrightarrow f$ is an antideriv.

<u>Examples</u>. ① X = tree

Antiderivs always exist.

$\Rightarrow H^1(X,G) = 0$.

② $X =$ 

$\Delta^1(X,G) \cong G$

No nonzero fn has antider.

$\Rightarrow H^1(X,G) \cong G$.

③ $X = \bigvee_{\alpha} S^1$

$H^1(X,G) = \prod_{\alpha} G$.

<u>More generally</u> X = any (oriented) graph

T = maximal tree/forest

E = edges outside T.

$\rightsquigarrow H^1(X,G) = \prod_{E} G$.

exercise. Hint: first consider fns $\equiv 0$ on T...

Show any other $f \in \Delta^1$ is cohom. to such a fn.

Two dimensions

$X = 2$-dim Δ-complex

$\delta : \Delta^1(X,G) \to \Delta^2(X,G)$

$\delta f([v_0,v_1,v_2]) = f(v_1v_2) - f(v_0v_2) + f(v_0v_1)$

Check that

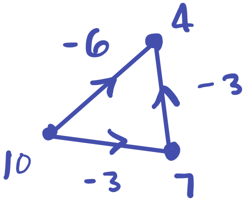
$$0 \xrightarrow{f} \Delta^0 \xrightarrow{\delta f} \Delta^1 \xrightarrow{\delta\delta f} \Delta^2 \to 0$$

is a chain complex:

$\delta\delta f([v_0,v_1,v_2]) = (f(v_2) - f(v_1)) - (f(v_2) - f(v_0)) + (f(v_1) - f(v_0))$

$= 0.$

If you hike/ski a loop, elevation change is 0.



What is a 1-cocycle? $(\ker \delta_1)$

$\delta f = 0 \Leftrightarrow f(v_0v_2) = f(v_0v_1) + f(v_1v_2)$

So: f is locally an antideriv.

on any triangle.

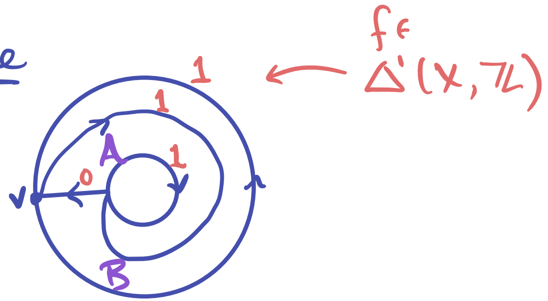
When is f a 1-coboundary? $(\mathrm{Im}\, \delta_0)$

When it's an antideriv.

So a nontrivial elt of $H^1(X)$ is a fn on edges that is locally, but not globally an antideriv.

Example

$X =$



Check f is a 1-cocycle

i.e. $\delta f = 0$.

$$\delta f(A) = 0$$
$$\delta f(B) = 0$$

But f is not in $\mathrm{im}\, \delta_0$:

Any value of $f(v)$ fails.