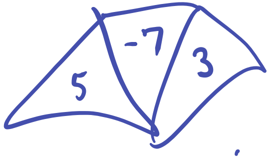
Mar 16

Cohomology

$\Delta^i(X)$ = functions from i-simplices of X to G.

= homoms $\Delta_i(X) \to G$

$i=2$



Claim $\longleftrightarrow$ Δ^i form a chain complex: $\operatorname{im}\delta_{-1} \subseteq \ker \delta_i$

$$\delta : \Delta^i(X,G) \to \Delta^{i+1}(X,G)$$
$$f \mapsto \delta f$$

for $f \in \Delta^i$, $\sigma = (i+1)$-simplex

$$\delta f(\sigma) = \sum (-1)^k f(\partial_k \sigma) \quad (*)$$

Same

c = ${}^{i+1}$chain $\quad f$ = ^{i}cochain

Claim: $\delta^2 = 0$.

Check: $\delta f(c) = f(\partial c) \quad (*)$

$$\delta\delta f(c) = \delta(f\partial c) = f\partial^2 c = f(0) = 0$$

$c = i+2$-chain

$H^*(X,G)$ = homology of this chain complex.

$$H^i(X,G) = \frac{\ker \delta_i}{\operatorname{im} \delta_{i-1}}$$

Two dimensions

$X = 2$-dim Δ-complex

$\delta : \Delta^1(X,G) \longrightarrow \Delta^2(X,G)$

$\delta f([v_0,v_1,v_2]) = f(v_1v_2) - f(v_0v_2) + f(v_0v_1)$

Check that

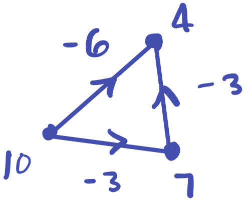
$$0 \longrightarrow \overset{f}{\Delta^0} \xrightarrow{\delta_0} \overset{\delta f}{\Delta^1} \xrightarrow{\delta_1} \overset{\delta\delta f}{\Delta^2} \longrightarrow 0$$

is a chain complex:

$$\delta\delta f([v_0,v_1,v_2]) = (f(v_2)-f(v_1)) - (f(v_2)-f(v_0)) + (f(v_1)-f(v_0))$$

$$= 0.$$

If you hike/ski a loop, elevation change is 0.



What is a 1-cocycle? ($\ker \delta_1$)

$\delta f = 0 \iff f(v_0v_2) = f(v_0v_1) + f(v_1v_2)$

So: f is locally a deriv.

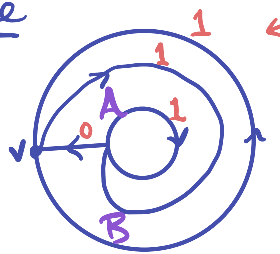
on any triangle.

When is f a 1-coboundary? ($\operatorname{Im} \delta_0$)

When it's a deriv.

So a nontrivial elt of $H^1(X)$ is a fn on edges that is locally, but not globally an a deriv.

Example

$X =$ 

$f \in \Delta^1(X, \mathbb{Z})$

Check f is a 1-cocycle

i.e. $\delta f = 0$.

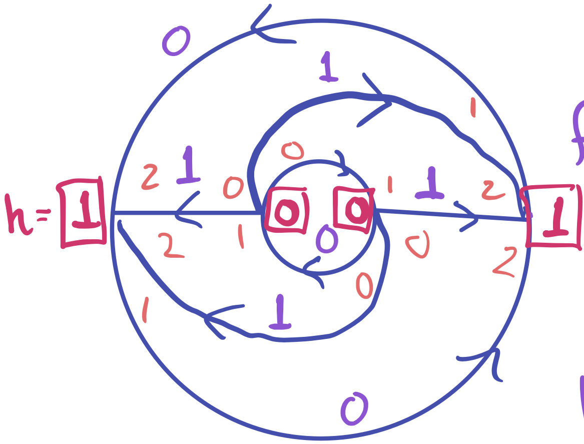
$\delta f(A) = 0$

$\delta f(B) = 0$

But f is not in $\operatorname{im} \delta_0$:

Any value of $f(v)$ fails.

Easier example



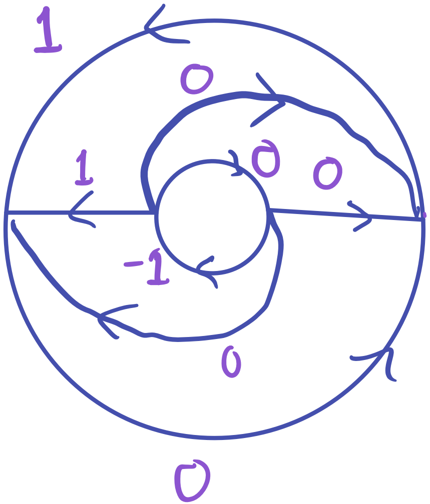
f a 1-cycle

It is a cobound.

It is δh

So: trivial elt of $H^1(X)$.

Another try.



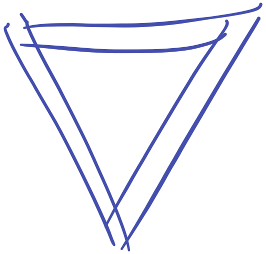
f a 1-cocycle.

It is not a cobound.

because the outside (and inside loops are nonzero).

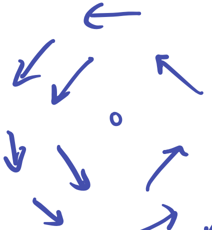
So: nontrivial elt of H^1.

Think about



Google: triangle optical illusion.

& DeRahm cohomology

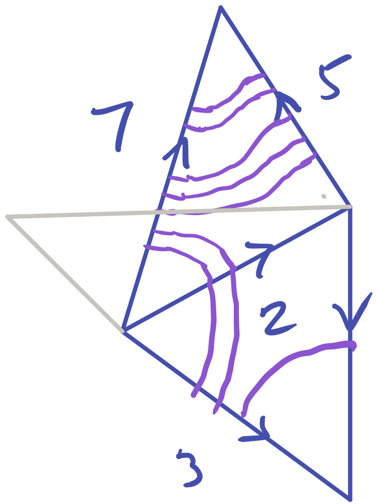


This vect field on $\mathbb{R}^2 \setminus 0$ is

__closed__: locally gradient

__not exact__: not globally a gradient

Geometric interpretation of 1-cocycles.



Cocycle condition
$(2+5=7)$

$\rightsquigarrow$ collection of arcs in each triangle

$\rightsquigarrow$ "curves" in X.